On the FPV property and the Monotone Polar of Representable Monotone Sets

Andrew Eberhard

RMIT University

WOMBAT 2018
Talk Summary

- What is a monotonicity?
- Representative Functions and Monotone sets
- Convex Functions and Monotonicity
- Why Maximality
- Maximality of Sums in Reflexive Spaces
- Monotonic Closure of Representable Monotone Set
- Generic Sum theorem Machinery
- Sum Theorem for FVP operators
What is Monotonicity?

- A convex pointed cone (i.e. $x \in C$ then $-x \notin C$) provides an order via $x \geq_C y$ iff $x - y \in C$. 

---

Andrew Eberhard (RMIT University) On the FPV property and the Monotone Polar WOMBAT 2018 3 / 33
What is Monotonicity?

- A convex pointed cone (i.e. $x \in C$ then $-x \notin C$) provides an order via $x \geq_C y$ iff $x - y \in C$.

- The polar cone is defined via the "duality pairing" of to Banach spaces $X$ and its dual $X^*$ (isometric to all continuous linear forms on $X$). In functional analysis this replaces the "inner product" used in finite dimensions.
What is Monotonicity?

- A convex pointed cone (i.e. $x \in C$ then $-x \notin C$) provides an order via $x \geq_C y$ iff $x - y \in C$.

- The polar cone is defined via the ”duality pairing” of to Banach spaces $X$ and its dual $X^*$ (isometric to all continuous linear forms on $X$). In functional analysis this replaces the ”inner product” used in finite dimensions.

- For $x^* \in X^*$ we have $x \mapsto \langle x, x^* \rangle$ a continuous linear functional on $X$ and for each $x \in X$ we have $x^* \mapsto \langle x, x^* \rangle$ a (weak*) continuous linear functional on $X^*$ i.e.

$$\langle \alpha_1 x_1 + \alpha_2 x_2, x^* \rangle = \alpha_1 \langle x_1, x^* \rangle + \alpha_2 \langle x_2, x^* \rangle$$

and

$$\langle \alpha_1 x_1^* + \alpha_2 x_2^*, x \rangle = \alpha_1 \langle x_1^*, x \rangle + \alpha_2 \langle x_2^*, x \rangle.$$
What is Monotonicity?

- A convex pointed cone (i.e. \( x \in C \) then \(-x \notin C\)) provides an order via \( x \geq_C y \iff x - y \in C\).

- The polar cone is defined via the "duality pairing" of to Banach spaces \( X \) and its dual \( X^* \) (isometric to all continuous linear forms on \( X \)). In functional analysis this replaces the "inner product" used in finite dimensions.

- For \( x^* \in X^* \) we have \( x \mapsto \langle x, x^* \rangle \) a continuous linear functional on \( X \) and for each \( x \in X \) we have \( x^* \mapsto \langle x, x^* \rangle \) a (weak*) continuous linear functional on \( X^* \) i.e.

\[
\langle \alpha_1 x_1 + \alpha_2 x_2, x^* \rangle = \alpha_1 \langle x_1, x^* \rangle + \alpha_2 \langle x_2, x^* \rangle
\]

\[
\text{and} \quad \langle \alpha_1 x_1^* + \alpha_2 x_2^*, x \rangle = \alpha_1 \langle x_1^*, x \rangle + \alpha_2 \langle x_2^*, x \rangle.
\]

- A Banach space \( X \) is reflexive if \( X^{**} = X \) i.e. the set of continuous linear forms on \( X^* \) is congruent to \( X \).
What is a Monotone Operator?

The polar cone is given by

\[ C^\circ = \{ x^* \in X^* \mid \langle x, x^* \rangle \leq 0, \text{ for all } x \in C \} \]
What is a Monotone Operator?

- Note $R^n_+ = - (R^n_+)^\circ$ in finite dimensions.

**Definition**

An operator $T : X \rightarrow X^*$ (possibly multi-valued) is called monotone iff for all $x^* \in Tx$ and $y^* \in Ty$ we have

$$\langle x - y, x^* - y^* \rangle \geq 0.$$
What is a Monotone Operator?

- Note $R^n_+ = -(R^n_+)^\circ$ in finite dimensions.
- If from $v \leq_C w$ it follows that $Tv \leq_{-C^\circ} Tw$ we have $v - w \in C$
  implies $Tw - Tv \in -C^\circ$ or

$$\langle Tw - Tv, w - v \rangle \geq 0.$$ 

Similarly if $w \leq_C v$ it follows that $Tw \leq_{-C^\circ} Tv$ we have $v - w \in C$
  implies $Tv - Tw \in -C^\circ$ or

$$\langle Tv - Tw, v - w \rangle \geq 0,$$ again.

**Definition**

An operator $T : X \rightarrow X^*$ (possibly multi-valued) is called monotone iff for all $x^* \in Tx$ and $y^* \in Ty$ we have

$$\langle x - y, x^* - y^* \rangle \geq 0.$$
Convex Functions

- The stock example of a "nice" monotone operator that can be multi-valued is the subgradient of a proper, extended real valued, closed convex function i.e. \( \text{epi}(f) := \{(x, \alpha) \mid \alpha \geq f(x)\} \) is closed, \( f > -\infty \) but can take values \( +\infty \) outside \( \text{dom } f := \{x \in X \mid f(x) < +\infty\} \).

**Definition**

Let \( f : X \to \mathbb{R}_{+\infty} \) be convex. The vector \( x^* \in X^* \) is a subgradient of \( f \) at \( \bar{x} \) if it satisfies the following inequality for all \( x \in X \)

\[
f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle \quad \text{(the subgradient inequality)}.\]

Denote by \( \partial f(\bar{x}) \) the set of all subgradients of \( f \) at \( \bar{x} \).
Convex Functions

- The stock example of a "nice" monotone operator that can be multi-valued is the subgradient of a proper, extended real valued, closed convex function i.e. $\text{epi}(f) := \{(x, \alpha) \mid \alpha \geq f(x)\}$ is closed, $f > -\infty$ but can take values $+\infty$ outside $\text{dom } f := \{x \in X \mid f(x) < +\infty\}$.

Definition

Let $f : X \to \mathbb{R}_{+\infty}$ be convex. The vector $x^* \in X^*$ is a subgradient of $f$ at $\bar{x}$ if it satisfies the following inequality for all $x \in X$

$$f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle \quad \text{(the subgradient inequality)}.$$ 

Denote by $\partial f(\bar{x})$ the set of all subgradients of $f$ at $\bar{x}$.

- Clearly $\partial f(\bar{x})$ is a convex set.
Convex Functions

- The stock example of a "nice" monotone operator that can be multi-valued is the subgradient of a proper, extended real valued, closed convex function \( i.e. \ epi(f) := \{(x, \alpha) \mid \alpha \geq f(x)\} \) is closed, \( f > -\infty \) but can take values \(+\infty\) outside

\[
\text{dom } f := \{x \in X \mid f(x) < +\infty\}.
\]

Definition

Let \( f : X \to \mathbb{R}_{+\infty} \) be convex. The vector \( x^* \in X^* \) is a subgradient of \( f \) at \( \bar{x} \) if it satisfies the following inequality for all \( x \in X \)

\[
f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle \quad (\text{the subgradient inequality}).
\]

Denote by \( \partial f(\bar{x}) \) the set of all subgradients of \( f \) at \( \bar{x} \).

- Clearly \( \partial f(\bar{x}) \) is a convex set.
- When the usual Frechet or Gateaux derivative \( \nabla f(\bar{x}) \) exists it is the unique subgradient.
Convex Functions is Monotone (and maximal)

- Write down the subgradient inequality twice (at $x$ and at $y$)

\[
  f(x) - f(y) \geq \langle y^*, x - y \rangle \quad \text{and} \quad f(y) - f(x) \geq \langle x^*, y - x \rangle
\]

adding implies

\[
  \langle x^* - y^*, x - y \rangle \geq 0.
\]
Convex Functions is Monotone (and maximal)

- Write down the subgradient inequality twice (at $x$ and at $y$)

\[ f(x) - f(y) \geq \langle y^*, x - y \rangle \quad \text{and} \quad f(y) - f(x) \geq \langle x^*, y - x \rangle \]

adding implies

\[ \langle x^* - y^*, x - y \rangle \geq 0. \]

- One can do this with a cycle of point $x_0, x_1, \ldots, x_n, x_{n+1} = x_0$ to get

\[ \sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \geq 0. \]
Convex Functions is Monotone (and maximal)

- Write down the subgradient inequality twice (at $x$ and at $y$)
  \[ f(x) - f(y) \geq \langle y^*, x - y \rangle \quad \text{and} \quad f(y) - f(x) \geq \langle x^*, y - x \rangle \]
  adding implies \[ \langle x^* - y^*, x - y \rangle \geq 0. \]
- One can do this with a cycle of point $x_0, x_1, \ldots, x_n, x_{n+1} = x_0$ to get
  \[ \sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \geq 0. \]
- Indeed $\partial f$ needs to be "Maximal" if $X$ is just a Banach space. That is, we can’t extend it any further as a monotone operator.
Convex Functions are "mostly" single valued

- There began an industry in the 1970-1980 (i.e. Kenderov, Phelps) around the linking of "Banach Geometric" properties, such as "rotundity of the dual norm" and properties of the duality mapping

\[ J(x) := \partial \frac{1}{2} \| x \|^2 \]

to the differentiability of convex functions on the Banach space in question.
Convex Functions are "mostly" single valued

- There began an industry in the 1970-1980 (i.e. Kenderov, Phelps) around the linking of "Banach Geometric" properties, such as "rotundity of the dual norm" and properties of the duality mapping

\[ J(x) := \partial \frac{1}{2} \|x\|^2 \]

to the differentiability of convex functions on the Banach space in question.

- Differentiainility corresponds to the single valuedness of the monotone operator \( T := \partial f \).
Convex Functions are “mostly” single valued

- There began an industry in the 1970-1980 (i.e. Kenderov, Phelps) around the linking of "Banach Geometric" properties, such as "rotundity of the dual norm" and properties of the duality mapping

\[ J(x) := \partial \frac{1}{2} \|x\|^2 \]

to the differentiability of convex functions on the Banach space in question.
- Differentiability corresponds to the single valuedness of the monotone operator \( T := \partial f \).
- An Asplund space (which includes all reflexive spaces) is defined to be those Banach spaces on which, equivalently:
Convex Functions are "mostly" single valued

- There began an industry in the 1970-1980 (i.e. Kenderov, Phelps) around the linking of "Banach Geometric" properties, such as "rotundity of the dual norm" and properties of the duality mapping

\[ J(x) := \partial \frac{1}{2} \|x\|^2 \]

to the differentiability of convex functions on the Banach space in question.

- Differentiability corresponds to the single valuedness of the monotone operator \( T := \partial f \).

- An Asplund space (which includes all reflexive spaces) is defined to be those Banach spaces on which, equivalently:
  
  - Every continuous convex function \( f \) on \( X \) is generically differentiable (generic means on a \( G_\delta \) dense subset)
Convex Functions are "mostly" single valued

- There began an industry in the 1970-1980 (i.e. Kenderov, Phelps) around the linking of "Banach Geometric" properties, such as "rotundity of the dual norm" and properties of the duality mapping
  
  \[ J(x) := \frac{1}{2} \|x\|^2 \]

  to the differentiability of convex functions on the Banach space in question.

- Differentiainlility corresponds to the single valuedness of the monotone operator \( T := \partial f \).

- An Asplund space (which includes all reflexive spaces) is defined to be those Banach spaces on which, equivalently:
  
  - Every continuous convex function \( f \) on \( X \) is generically differentiable (generic means on a \( G_\delta \) dense subset)
  - Every (maximal) monotone operator is single-values on a generic subset of its domain.
Why Maximality

- A lot of work in infinite dimension revolves around finite approximation and an easy consequence of maximality is the demi-closed property:

\[ x_n^* \in T(x_n), \quad x_n^* \rightarrow_w x^*, x_n \rightarrow_s x \implies x^* \in T(x). \]
Why Maximality

- A lot of work in infinite dimension revolves around finite approximation and an easy consequence of maximality is the demi-closed property:

\[ x^*_n \in T(x_n), \quad x^*_n \to_w x^*, x_n \to_s x \implies x^* \in T(x). \]

- This is just what is needed for approximation methods. This also arise in optimization algorithm.
Why Maximality

- A lot of work in infinite dimension revolves around finite approximation and an easy consequence of maximality is the demi-closed property:

\[ x_n^* \in T(x_n), \quad x_n^* \rightharpoonup x^*, \quad x_n \rightharpoonup x \implies x^* \in T(x). \]

- This is just what is needed for approximation methods. This also arises in optimization algorithm.

- Sums of maximal monotone operators arise frequently and we would like to know if they are maximal?
Why Maximality

- A lot of work in infinite dimension revolves around finite approximation and an easy consequence of maximality is the demi-closed property:

\[ x_n^* \in T(x_n), \quad x_n^* \rightharpoonup w x^*, x_n \rightharpoonup_s x \implies x^* \in T(x). \]

- This is just what is needed for approximation methods. This also arise in optimization algorithm.

- Sums of maximal monotone operators arise frequently and we would like to know if they are maximal?

- Outside of reflexive spaces question regarding maximality and closeness are much more difficult due to total failure of any kind of joint continuity of the duality mapping \((x, x^*) \mapsto \langle x, x^* \rangle\) with respect to any topology compatible with \(s \times w^*\) convergence (and duality).
Maximality of Sums

As we have seen many applications require maximality of a sum of monotone operators. This issue was resolved in Reflexive spaces by Terry Rockafellar, ”On the maximality of a Sum of Nonlinear Monotone Operators”, Trans. Am. Math. Soc., 159, 81-99, 1970.

**Theorem (Rockafellar’s Sum Theorem)**

Suppose that $S$ and $T$ are maximal monotone operators on a reflexive Banach space. Suppose that

$$\text{int dom } (S) \cap \text{dom } (T) \neq \emptyset.$$

Then $S + T$ is maximal monotone.

Maximality is the hard part to prove, monotonicity is trivial: Suppose $x^* \in T(x)$ and $y^* \in S(x)$ then $x^* + y^* \in S(x) + T(x) = (S + T)(x)$ and for $u^* \in T(u)$ and $v^* \in S(u)$, using monotonicity of $S$ and $T$,

$$\langle x^* + y^* - (u^* + v^*), x - u \rangle = \langle x^* - u^*, x - u \rangle + \langle y^* - v^*, x - u \rangle \geq 0.$$
Consequences of a Sum Theorem

- There are many consequences of a sum theorem when it holds, the domain \( \text{dom } T \) is "nearly convex" i.e. \( \text{dom } \overline{T} \) is convex.

Corollary (Convex Closure, Simons)

Assume the sum theorem. Suppose \( T \) is maximal monotone then \( T \) is of type \((FPV)\). In particular, \( \text{dom}(T) \) has a convex closure.
Consequences of a Sum Theorem

- There are many consequences of a sum theorem when it holds, the domain \( \text{dom} \, T \) is "nearly convex" i.e. \( \overline{\text{dom} \, T} \) is convex.

- Martínez-Legaz and Svaiter introduced the monotone polar of monotone set \( T \subseteq X \times X^* \) which is denoted by

\[
T^m := \{(x, x^*) \in X \times X^* \mid \langle x - y, x^* - y^* \rangle \geq 0, \forall (y, y^*) \in T\}.
\]

Corollary (Convex Closure, Simons)

Assume the sum theorem. Suppose \( T \) is maximal monotone then \( T \) is of type (FPV). In particular, \( \text{dom}(T) \) has a convex closure.
Consequences of a Sum Theorem

- There are many consequences of a sum theorem when it holds, the domain \( \text{dom} \, T \) is "nearly convex" i.e. \( \overline{\text{dom} \, T} \) is convex.

- Martínez-Legaz and Svaiter introduced the monotone polar of monotone set \( T \subseteq X \times X^* \) which is denoted by

\[
T^\mu := \{(x, x^*) \in X \times X^* \mid \langle x - y, x^* - y^* \rangle \geq 0, \forall (y, y^*) \in T\}.
\]

- \( T \) is of type (FPV): \( \forall V \subseteq X \) with \( \text{dom} \, T \cap V \neq \emptyset \) the following holds: if \( x \in V \) and \((x, x^*) \in (T|_V)^\mu \) implies \( x^* \in T(x) \).

Corollary (Convex Closure, Simons)

Assume the sum theorem. Suppose \( T \) is maximal monotone then \( T \) is of type (FPV). In particular, \( \text{dom}(T) \) has a convex closure.
Representative Functions

- Couple $X \times X^*$ with $X^* \times X$ using $\langle (z, z^*), (x^*, x) \rangle = \langle z, x^* \rangle + \langle x, z^* \rangle$ and $\| (z, z^*) \|^2 = \| z \|^2 + \| z^* \|^2$.

**Definition**

We call a proper convex function $f$ is representative when $f(y, y^*) \geq \langle y, y^* \rangle$ for all $(y, y^*) \in X \times X^*$ and it represents $M_f := \{(x, x^*) \in X \times X^* \mid f(x, x^*) = \langle x, x^* \rangle\}$.

**Theorem (Fitzpatrick/Penot?)**

*If f is representative then $M_f$ is a monotone set.*

**Proof:** Let $(x, x^*), (y, y^*) \in M_f$ then $\langle x - y, x^* - y \rangle \geq 0$ follows from

$$\frac{1}{2} \langle x, x^* \rangle + \frac{1}{2} \langle y, y^* \rangle = \frac{1}{2} f(x, x^*) + \frac{1}{2} f(y, y^*) \geq f\left(\frac{1}{2} (x, x^*) + \frac{1}{2} (y, y^*)\right) \geq \langle \frac{1}{2} (x + y), \frac{1}{2} (x^* + y^*) \rangle = \frac{1}{4} \langle x, x^* \rangle + \frac{1}{4} \langle x, y^* \rangle + \frac{1}{4} \langle y, x^* \rangle + \frac{1}{4} \langle y, y^* \rangle.$$
The Fitzpatrick Function

- Define the ‘transpose’ operator $\dagger$: $(x^*, x) \leftrightarrow (x, x^*)$ and $c_T(\cdot, \cdot) := \delta_T(\cdot, \cdot) + \langle \cdot, \cdot \rangle$, ($\delta_T$ the indicator of the graph of $T$).

Fitzpatrick showed that

$$P_T = F_T^*\dagger \text{ and } F_T = c_T^{*\dagger} \text{ defined on } (X, X^*)$$

are representative when $T$ is maximal. Indeed, $P_T$ is the largest under the pointwise order and $F_T$ the smallest. As a consequence proofs were reduced from dozens of pages to half pages.

Figure: Simon Fitzpatrick and Regina Burachik
The Reflexive Case

In a reflexive space there is a beautiful characterisation of maximal monotonicity in terms of representative functions:

Denote \( M_h^\leq := \{(x, x^*) \in X \times X^* \mid h(x, x^*) \leq \langle x, x^* \rangle \} \) then \( T^\mu = M_h^\leq \).

Theorem (Burachik and Svaiter)

Suppose \( X \) is a reflexive Banach space and \( h : X \times X^* \to \mathbb{R}_{+\infty} \) be a convex lower semi-continuous function. Suppose that

\[
\forall (x, x^*) \in X \times X^* \quad h(x, x^*) \geq \langle x, x^* \rangle, \quad h^*(x, x^*) \geq \langle x, x^* \rangle.
\]

Then \( T := M_h^\leq \) is maximal monotone and \( h \in R(T) \).
The Non-reflexive Case

- So as to respect the basic duality relationships for conjugation on $X \times X^*$ paired with $X^* \times X$, with the later endowed with the $w^* \times s$ topology.
The Non-reflexive Case

- So as to respect the basic duality relationships for conjugation on $X \times X^*$ paired with $X^* \times X$, with the later endowed with the $w^* \times s$ topology.
- Denote

$$f^{*\dagger}(x, x^*) := f^*(x^*, x) = \sup_{(z, z^*) \in X \times X^*} \{\langle (x, x^*) , (z, z^*) \rangle - f(z, z^*) \}$$

is the transpose conjugate of $f$. 

Andrew Eberhard (RMIT University) On the FPV property and the Monotone Polar WOMBAT 2018 15 / 33
The Non-reflexive Case

- So as to respect the basic duality relationships for conjugation on $X \times X^*$ paired with $X^* \times X$, with the later endowed with the $w^* \times s$ topology.

- Denote

\[
f^* \dagger (x, x^*) := f^* (x^*, x) = \sup_{(z, z^*) \in X \times X^*} \{ \langle (x, x^*), (z, z^*) \rangle - f(z, z^*) \}
\]

is the transpose conjugate of $f$.

- We say $T$ is representable when there exists $f \in R(T) := \{ f \in [\mathcal{F}_T, \mathcal{P}_T] \mid f \geq \langle \cdot, \cdot \rangle \}$ with $T = M_f := M_f^\leq$ when $f$ is representative.
The Non-reflexive Case

- So as to respect the basic duality relationships for conjugation on $X \times X^*$ paired with $X^* \times X$, with the later endowed with the $w^* \times s$ topology.

- Denote

$$f^* \dagger (x, x^*) := f^*(x^*, x) = \sup_{(z, z^*) \in X \times X^*} \{ \langle (x, x^*), (z, z^*) \rangle - f(z, z^*) \}$$

is the transpose conjugate of $f$.

- We say $T$ is representable when there exists $f \in R(T) := \{ f \in [\mathcal{F}_T, \mathcal{P}_T] \mid f \geq \langle \cdot, \cdot \rangle \}$ with $T = M_f := M_f^\leq$ when $f$ is representative.

- It is well known that when $f \in R(T)$ then $f \in [\mathcal{F}_T, \mathcal{P}_T] = \{ g \in PC(X, X^*) \mid \mathcal{F}_T \leq g \leq \mathcal{P}_T \}$, where the partial order is pointwise.
The Non-reflexive Case

- So as to respect the basic duality relationships for conjugation on $X \times X^*$ paired with $X^* \times X$, with the later endowed with the $w^* \times s$ topology.
- Denote

$$f^{\dagger} (x, x^*) := f^* (x^*, x) = \sup_{(z, z^*) \in X \times X^*} \{\langle (x, x^*), (z, z^*) \rangle - f(z, z^*)\}$$

is the transpose conjugate of $f$.
- We say $T$ is representable when there exists $f \in R(T) := \{f \in \mathcal{F}_T, \mathcal{P}_T \mid f \geq \langle \cdot, \cdot \rangle\}$ with $T = M_f := M_f^\subseteq$ when $f$ is representative.
- It is well known that when $f \in R(T)$ then $f \in [\mathcal{F}_T, \mathcal{P}_T] = \{g \in PC(X, X^*) \mid \mathcal{F}_T \leq g \leq \mathcal{P}_T\}$, where the partial order is pointwise.
- Recall $bR(T) := \{f \in [\mathcal{F}_T, \mathcal{P}_T] \mid f^{\dagger} \geq f \geq \langle \cdot, \cdot \rangle\}$ are the bigger–conjugate representative functions with $T \subseteq M_f \subseteq T^\mu$. 

Andrew Eberhard (RMIT University) On the FPV property and the Monotone Polar WOMBAT 2018 15 / 33
Monotonic Closure

- One always has $T^{\mu\mu\mu} = T^{\mu}$ and if $T$ is monotone then $T \subseteq T^{\mu}$ and $T \rightharpoonup T^{\mu}$ is a polarity and as a consequence $A \subseteq B$ implies $A^\mu \supseteq B^\mu$, $T \subseteq T^{\mu\mu}$ and $(A \cup B)^\mu = A^\mu \cap B^\mu$ (for any sets $A, B \subseteq X \times X^*$).

Proposition (Martinez-Lagaz – Svaiter)

The following are equivalent to $T : X \rightharpoonup X^*$ being monotone:

1. $T \subseteq T^\mu$,
2. $T^{\mu\mu} \subseteq T^\mu$,
3. $T^{\mu\mu}$ is monotone (with $T \subseteq T^{\mu\mu}$).

We have $T$ maximal monotone iff $T = T^\mu$ (or $T$ monotone and $T \supseteq T^\mu$).

Moreover denoting $\mathbf{M}(T) := \{ B \subseteq X \times X^* \mid B$ is maximal monotone, $T \subseteq B \}$ then when $T$ is monotone we have:

$T^\mu = \bigcup_{B \in \mathbf{M}(T)} B$ and $T^{\mu\mu} = \bigcap_{B \in \mathbf{M}(T)} B$. 
Monotonic Closure

- One always has $T^{\mu\mu\mu} = T^\mu$ and if $T$ is monotone then $T \subseteq T^\mu$ and $T \mapsto T^\mu$ is a polarity and as a consequence $A \subseteq B$ implies $A^\mu \supseteq B^\mu$, $T \subseteq T^{\mu\mu}$ and $(A \cup B)^\mu = A^\mu \cap B^\mu$ (for any sets $A, B \subseteq X \times X^*$).
- We shall call $T^{\mu\mu}$ the monotonic closure of $T$.

**Proposition (Martinez-Lagaz – Svaiter)**

The following are equivalent to $T : X \rightrightarrows X^*$ being monotone:

1. $T \subseteq T^\mu$,
2. $T^{\mu\mu} \subseteq T^\mu$,
3. $T^{\mu\mu}$ is monotone (with $T \subseteq T^{\mu\mu}$).

We have $T$ maximal monotone iff $T = T^\mu$ (or $T$ monotone and $T \supseteq T^\mu$). Moreover denoting 

$\mathcal{M}(T) := \{ B \subseteq X \times X^* \mid B \text{ is maximal monotone}, T \subseteq B \}$ then when $T$ is monotone we have:

$T^\mu = \bigcup_{B \in \mathcal{M}(T)} B$ and $T^{\mu\mu} = \bigcap_{B \in \mathcal{M}(T)} B$. 
In a reflexive space all $f \in bR(T)$ represent a maximal monotone operators where $T := M_f = M_{f^*}$.
In a reflexive space all $f \in bR(T)$ represent a maximal monotone operators where $T := M_f = M_{f^*}$

The main idea studied here is that under the assumption that $M_h$ for $h \in bR(T)$ is not maximal, then there exists points $(x, x^*) \in (M_h)^\mu \cap (M_h)^c$ (and so $\{(x, x^*)\}^\mu \supset M_h$).
In a reflexive space all \( f \in bR(T) \) represent a maximal monotone operators where \( T := M_f = M_{f^*} \).

The main idea studied here is that under the assumption that \( M_h \) for \( h \in bR(T) \) is not maximal, then there exists points \( (x, x^*) \in (M_h)^\mu \cap (M_h)^c \) (and so \( \{(x, x^*)\}^\mu \supseteq M_h \)).

Indeed if this was not the case, then we would necessarily have \( (M_h)^\mu \subseteq M_h \subseteq (M_h)^\mu \) implying \( M_h \) is maximal.
In a reflexive space all $f \in bR(T)$ represent a maximal monotone operators where $T := M_f = M_{f^*}$

The main idea studied here is that under the assumption that $M_h$ for $h \in bR(T)$ is not maximal, then there exists points $(x, x^*) \in (M_h)'^* \cap (M_h)^c$ (and so $\{(x, x^*)\}^\mu \supset M_h$).

Indeed if this was not the case, then we would necessarily have $(M_h)'^* \subseteq M_h \subseteq (M_h)'^*$ implying $M_h$ is maximal.

This raises the question of the existence of a representative function for the monotone set $M_h \cup \{(x, x^*)\}$? We need to include $\{(x, x^*), \langle x, x^* \rangle\}$ into the graph of the new representative function $g$, via convexification.
In a reflexive space all \( f \in bR(T) \) represent a maximal monotone operators where \( T := M_f = M_{f^*} \).

The main idea studied here is that under the assumption that \( M_h \) for \( h \in bR(T) \) is not maximal, then there exists points \( (x, x^*) \in (M_h)^u \cap (M_h)^c \) (and so \( \{(x, x^*)\}^u \supseteq M_h \)).

Indeed if this was not the case, then we would necessarily have \( (M_h)^u \subseteq M_h \subseteq (M_h)^u \) implying \( M_h \) is maximal.

This raises the question of the existence of a representative function for the monotone set \( M_h \cup \{(x, x^*)\} \)? We need to include \( \{(x, x^*), \langle x, x^* \rangle\} \) into the graph of the new representative function \( g \), via convexification.

In effect we are seeking a representative function for a monotone extension of \( M_h \) to include the new point. As a consequence we quickly obtain that when \( h \in bR(T) \) we have \( M_h \) monotonically closed.
Suppose $h \in bR(T)$ and that $M_h$ is not maximal. Let $(x, x^*) \notin M_h$ and consider a convex minorant defined by

$$
g := \text{co}[h, \delta \{(x, x^*)\} + \langle x, x^* \rangle] \leq h$$

(1)
To study the problem of maximality we use the construction in (1) to define a minorising convex function \( g \in b(T) \) whenever there exists a point \((x, x^*) \in (M_h)^{\mu} \cap (M_h)^{c}\).
To study the problem of maximality we use the construction in (1) to define a minorising convex function $g \in b(T)$ whenever there exists a point $(x, x^*) \in (M_h)^\mu \cap (M_h)^c$.

The challenge is to show that there is a choice of $(x, x^*) \in (M_h)^\mu \cap (M_h)^c$ for which $M_g$ is monotone.
To study the problem of maximality we use the construction in (1) to define a minorising convex function \( g \in b(T) \) whenever there exists a point \((x, x^*) \in (M_h)^\mu \cap (M_h)^c\).

The challenge is to show that there is a choice of \((x, x^*) \in (M_h)^\mu \cap (M_h)^c\) for which \( M_g^{\leq} \) is monotone.

As we clearly have \( M_h \cup \{(x, x^*)\} \subseteq M_g^{\leq} \) this will only be the case if \( M_g^{\leq} \subseteq (M_h \cup \{(x, x^*)\})^{\mu} \).
To study the problem of maximality we use the construction in (1) to define a minorising convex function $g \in b(T)$ whenever there exists a point $(x, x^*) \in (M_h)^\mu \cap (M_h)^c$.

The challenge is to show that there is a choice of $(x, x^*) \in (M_h)^\mu \cap (M_h)^c$ for which $M_g^\leq$ is monotone.

As we clearly have $M_h \cup \{(x, x^*)\} \subseteq M_g^\leq$, this will only be the case if $M_g^\leq \subseteq (M_h \cup \{(x, x^*)\})^\mu$.

Lemma

Suppose $h \in bR(T)$ and $(x, x^*) \in (M_h)^\mu \cap (M_h)^c$. Let $g$ be the function constructed via (1). Then when $M_g^\leq$ is monotone we have

$$M_g^\leq \subseteq \{(x, x^*)\}^\mu.$$  \hspace{1cm} (2)

Moreover (2) is equivalent to $M_g^\leq \subseteq (M_h \cup \{(x, x^*)\})^\mu$. 
(Monotone Extensions and Maximality) Let \( h \in bR(T) \) and let \( g \) be defined as in (1) using \( (x, x^*) \in (M_h)^\mu \cap (M_h)^c \).

1. When \( (z, z^*) \in \{(x, x^*)\}^\mu \) then \( g(z, z^*) \geq \langle z, z^* \rangle \).
2. When (2) holds, then \( g \in bR(M_g) \) and \( g \geq F_{M_g} = F_{M_g^<>} \).
3. If \( F_{M_h} \geq \langle \cdot, \cdot \rangle \), then \( h^{s\times w^*} \in bR(T) \).
4. The function

\[
f(z, z^*) := \max \{ g(z, z^*), \langle z, x^* \rangle + \langle z^*, x \rangle - \langle x, x^* \rangle \} \quad (3)
\]

is always representative and \( M_f = M_{g^<} \cap \{(x, x^*)\}^\mu \) is a monotone extension of \( M_h \cup \{(x, x^*)\} \).

5. If \( M_h \) is not maximal, then there cannot exist \( (x, x^*) \in (M_h)^\mu \cap (M_h)^c \) such that (2) holds for \( g \).
As a consequence we may prove the following theorems.

**Theorem**

A monotone operator $M$ is monotonically closed iff $M$ is representable i.e. there exists $h \in bR(M)$ such that $M = M_h$. That is we always have

$$(M_h)^{\mu\mu} = M_h.$$ 

We say that an operator $T : X \rightrightarrows X^*$ is **completely closed** iff 1). for any bounded net $(x_\alpha, x_\alpha^*) \to (x, x^*)$ with $(x_\alpha, x_\alpha^*) \in T$ we have $(x, x^*) \in T$. 2). whenever $(x_\alpha, x_\alpha^*) \in T$ with $x_\alpha \to x \in \text{dom } T$ and $\|x_\alpha^*\| \to \infty$ we have $\exists t_\alpha \to 0^+$ such that along some subnet $t_\alpha x_\alpha^* \rightharpoonup x^* \in 0^+ T(x)$.

**Theorem**

*(Complete Closure of Representable Operators)* If have $M = M_h$ for $h \in bR(T)$ then $M$ is completely closed.
Sum Theorem

Then the partial inf-convolutions are the functions defined on $X \times Y$ by

\[
F_1 \Box_1 F_2 : (x, y) \mapsto \inf_{u \in X} [F_1(u, y) + F_2(x - u, y)]
\]

\[
F_1 \Box_2 F_2 : (x, y) \mapsto \inf_{v \in Y} [F_1(x, y - v) + F_2(x, v)].
\]

Theorem (Generic Sum Theorem tool)

Let $X$ be a nonzero, real Banach space. Let $T_1$ and $T_2$ be maximal monotone operators from $X$ to $X^*$. Suppose $F_i \in bR(T_i)$ are representative functions for $T_i$, for $i = 1, 2$ and

\[
\bigcup_{\lambda > 0} \lambda \left[ P_X \text{dom } F_1 - P_X \text{dom } F_2 \right]
\]

is a closed subspace of $X$. \hspace{1cm} (4)

Then $F := F_1 \Box_2 F_2$ gives a bigger–conjugate representative function for $T_1 + T_2$ (i.e. $F \in bR(T_1 + T_2)$) for which $M_F = T_1 + T_2$. Consequently $T_1 + T_2$ is representable and hence monotonically closed i.e. $(T_1 + T_2)^{\mu\mu} = T_1 + T_2$. Note (4) is implied by $\text{dom } T_1 \cap \text{int dom } T_2 \neq \emptyset$. 

Andrew Eberhard (RMIT University) 
On the FPV property and the Monotone Polar 
WOMBAT 2018 22 / 33
The FPV property for $T$ can now be written as $(T|_U)^\mu |_U \subseteq T$ for any open convex neighbourhood $U$ with $U \cap \text{dom } T \neq \emptyset$. 
The FPV property for $T$ can now be written as $(T|_U)^\mu |_U \subseteq T$ for any open convex neighbourhood $U$ with $U \cap \text{dom } T \neq \emptyset$.

Let $\widetilde{T}_1 := T_1 \cap \text{dom } T_2 \times X^* = (T_1)|_{\text{dom } T_2}$.
The FPV property for $T$ can now be written as $(T|U)^\mu|_U \subseteq T$ for any open convex neighbourhood $U$ with $U \cap \text{dom } T \neq \emptyset$.

Let $\tilde{T}_1 := T_1 \cap [\text{dom } T_2 \times X^*] = (T_1)|_{\text{dom } T_2}$.

Given an $x \in \text{dom } T_1 \cap \text{int dom } T_2$ and an open convex set $U$ with $x \in U \subseteq \text{int dom } T_2$, we have by the properties of polarity:

$$(x, x^*) \in \left(\tilde{T}_1\right)^\mu = (T_1|_{\text{dom } T_2})^\mu \subseteq \left(T_1|_{U \cap \text{dom } T_2}\right)^\mu = (T_1|_U)^\mu$$

$$\implies (x, x^*) \in T.$$
The FPV property for $T$ can now be written as $(T | U)^\mu | U \subseteq T$ for any open convex neighbourhood $U$ with $U \cap \text{dom } T \neq \emptyset$.

Let $\tilde{T}_1 := T_1 \cap [\text{dom } T_2 \times X^*] = (T_1) |_{\text{dom } T_2}$.

Given an $x \in \text{dom } T_1 \cap \text{int } \text{dom } T_2$ and an open convex set $U$ with $x \in U \subseteq \text{int } \text{dom } T_2$, we have by the properties of polarity:

$$(x, x^*) \in (\tilde{T}_1)^\mu = (T_1 |_{\text{dom } T_2})^\mu \subseteq (T_1 |_{U \cap \text{dom } T_2})^\mu = (T_1 | U)^\mu$$

$\implies (x, x^*) \in T.$

This observation motivates us to study the operator $(\tilde{T}_1)^\mu |_{\text{dom } T_2}$. 

Andrew Eberhard (RMIT University) 
On the FPV property and the Monotone Pola 
WOMBAT 2018 23 / 33
The FPV property for $T$ can now be written as $(T|_U)^\mu |_U \subseteq T$ for any open convex neighbourhood $U$ with $U \cap \text{dom } T \neq \emptyset$.

Let $\sim T_1 := T_1 \cap [\text{dom } T_2 \times X^*] = (T_1)|_{\text{dom } T_2}$.

Given an $x \in \text{dom } T_1 \cap \text{int dom } T_2$ and an open convex set $U$ with $x \in U \subseteq \text{int dom } T_2$, we have by the properties of polarity:

$$(x, x^*) \in (\sim T_1)^\mu = (T_1|_{\text{dom } T_2})^\mu \subseteq (T_1|_{U \cap \text{dom } T_2})^\mu = (T_1|_U)^\mu$$

$\implies (x, x^*) \in T$.

This observation motivates us to study the operator $(\sim T_1)^\mu |_{\text{dom } T_2}$.

Ultimately we will consider the case when $T_2 = N_C$ where $\text{dom } T_1 \cap \text{int } C \neq \emptyset$ and $C$ is a closed convex set.
In the following we denote by \((T_1 + T_2) : X \rightarrow X^*\) the operator whose graph consists of all points monotonically related to the graph of \(T_1 + T_2\).

**Theorem**

* (Polar Sum Theorem) Suppose \(T_1\) and \(T_2\) are monotone operators from \(X\) to \(X^*\). Suppose \(T_2\) is conic valued and that \(0 \in (T_2) (x)\) for all \(x \in \text{dom}(T_2)\). Then

\[
(T_1 + T_2) (x) = \left( \tilde{T}_1 \right) (x) + (T_2) (x)
\]

for \(x \in \text{dom}(T_1 + T_2)\)

\[
= \text{dom} \left( \tilde{T}_1 \right) \cap \text{dom} (T_2)
\]

where \(\tilde{T}_1 := T_1 \cap \text{[dom } T_2 \times X^* \} = (T_1) |_{\text{dom } T_2}.\)
Theorem

*(Sum Maximality Theorem)* Suppose $T_1$ and $T_2$ be monotone operators from $X$ to $X^*$ with $\text{dom } T_1 \cap \text{dom } T_2 \neq \emptyset$. Suppose $T_2$ is conic valued with also $0 \in T_2^\mu (x)$ for all $x \in \text{dom } T_2^\mu$ and let $\tilde{T}_1 := T_1|_{\text{dom } T_2}$ and

$$M := \left( \tilde{T}_1 \right)^\mu \big|_{\text{dom } T_2^\mu}.$$  

Then

$$M^\mu \subseteq (T_1 + T_2)^{\mu\mu} \subseteq M^{\mu\mu}$$

so $M^\mu$ is monotone, and when $M$ is monotone then $(T_1 + T_2)^{\mu\mu}$ is maximal (i.e. $(T_1 + T_2)^\mu$ is pre-maximal) and hence $T_1 + T_2$ is maximal if monotonically closed.

Note that the key assumptions that establishes maximality is the monotonicty of the set $M := \left( \tilde{T}_1 \right)^\mu \big|_{\text{dom } T_2}$. 

When $T_2 = N_{\overline{U}}(u) := (\overline{\text{cone}(U - u)})^\circ$ we obtain identities.

**Lemma**

(Polarity Lemma) Let $T$ be maximal monotone from $X$ to $X^*$ and $U$ a open convex set with $\text{dom } T \cap U \neq \emptyset$. Then

$$
(T + N_{\overline{U}})^\mu = (T|_{\overline{U}})^\mu|_{\overline{U}} + N_{\overline{U}} = (T|_{\overline{U}})^\mu|_{\overline{U}}
$$

(6)

and

$$(T + N_{\overline{U}})^{\mu\mu} = T + N_{\overline{U}} = \left((T|_{\overline{U}})^\mu|_{\overline{U}} + N_{\overline{U}}\right)^\mu = \left((T|_{\overline{U}})^\mu|_{\overline{U}}\right)^\mu.
$$

(7)

When $T$ is of type FPV then

$$
\emptyset \neq \text{dom } T \cap U = \text{dom } (T|_{\overline{U}})^\mu \cap U
$$

(8)

and

$$
\overline{\text{dom } T \cap U} = \text{dom } (T|_{\overline{U}})^\mu \cap U \quad \text{a convex set.}
$$

(9)

Suppose in addition that $X$ admits a strictly convex re-norm. Then we have

$$
\text{dom } T \cap \overline{U} = \text{dom } (T|_{\overline{U}})^\mu \cap \overline{U}.
$$

(10)
The following is an important tool for the development of the sum theorem.

Essentially we need to know when \((T|_{U})^{\mu}|_{U}\) is a monotone set. The implication (11) suffices as it contains this set in another monotone set.

**Proposition**

Suppose \(T: X \rightrightarrows X^*\) is maximal monotone of type FPV and \(U \subseteq X\) is an open convex set such that \(U \cap \text{dom} \, T \neq \emptyset\). Then we have

\[
(u, u^*) \in (T|_{U})^{\mu}|_{U} \implies u^* + x^* \in \overline{T(u) + N_{U}(u)}, \text{ for all } x^* \in N_{U}(u).
\]

when one of the following additional assumptions hold:

1. **The convex set \(\overline{U}\) is strictly convex (i.e. for any \((u, u^*), (v, v^*) \in \text{bd} \, \overline{U}\) we have \(\lambda \, (u, u^*) + (1 - \lambda) \, (v, v^*) \in U\) for some \(\lambda \in (0, 1)\)).**

2. **The space \(X\) admits a strictly convex re-norm.**

In particular \((T|_{U})^{\mu}|_{U}(u) \subseteq \overline{T(u) + N_{U}(u)}\) for all \(u\).
In view of Theorem 11 we have the following result which appears to not be a consequence of the known sum theorems for operators of type FPV where additional structural assumptions are made on \( \text{dom } T \).

**Theorem (Normal Cone Sums)**

Suppose \( X \) is a real Banach space that admits a strictly convex re-norm, \( T \) is maximal monotone of type FPV and \( C \) is closed and convex with

\[
\text{dom } T \cap \text{int } C \neq \emptyset. \quad (12)
\]

Then \( T + N_C \) is maximal monotone.
In view of Theorem 11 we have the following result which appears to not be a consequence of the known sum theorems for operators of type FPV where additional structural assumptions are made on \( \text{dom } T \).

We note that the existence of a strictly convex re-norm is a very weak assumption and such space contain reflexive Banach space, separable Banach spaces, WCG spaces, duals of Asplund spaces due to the duality between Gateau differentiability and strict convexity of the dual norm.

**Theorem (Normal Cone Sums)**

Suppose \( X \) is a real Banach space that admits a strictly convex re-norm, \( T \) is maximal monotone of type FPV and \( C \) is closed and convex with

\[
\text{dom } T \cap \text{int } C \neq \emptyset. \tag{12}
\]

Then \( T + N_C \) is maximal monotone.
We may now obtain a characterisation of the FPV property.

**Corollary**

*Suppose* $X$ is a real Banach space with a strictly convex re-norm, $T$ is maximal monotone and $C \subseteq X$ is closed, convex with (12) holding. Then $T + N_C$ is maximal monotone iff $T$ is of type FPV.*

It is still possible that this basic sum theorem fails for some pathological operators. This sum theorem issue can now be resolved by the resolution of the following question, which to the authors knowledge still remain open.

**Questions:** Does there exists a maximal monotone operator on a real Banach space $X$ which is not of type FPV?

Does there exists FPV maximal monotone operators that fail to admit a sum theorem with a normal cone of a (non)strictly convex set outside the strictly convex re-normable spaces?
We finish by observing the following.

**Theorem**

If $T$ is maximal monotone on a real Banach space $X$ that admits a strictly convex re-norm. Then consider the following are all equivalent:

1. **$T$ is of type FPV i.e.** for all open convex sets $U$ we have
   \[(y, y^*) \in (T|_U)^\mu |_U \implies (y, y^*) \in T.\]

2. **Whenever $U \subseteq X$ is an open, convex set such that $U \cap \text{dom } T \neq \emptyset$, then we have**
   \[(y, y^*) \in (T|_U)^\mu |_U \implies (y, y^* + x^*) \in T + N_{\overline{U}}, \text{ for any } x^* \in N_{\overline{U}}(y). \tag{13}\]

3. **Whenever $C$ is closed convex with $\text{dom } T \cap \text{int } C \neq \emptyset$. Then we have**
   \[(T|_C)^\mu |_C \subseteq T + N_C.\]

4. **When $C$ is a closed, convex set with $\text{dom } T \cap \text{int } C \neq \emptyset$ then $T + N_C$ is maximal monotone.**
Thank You!


