On the FPV property and the Monotone Polar of Representable Monotone Sets

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Talk Summary

- What is a monotonicity?
- Representative Functions and Monotone sets
- Convex Functions and Monotonicity
- Why Maximality
- Maximality of Sums in Reflexive Spaces
- Monotonic Closure of Representable Monotone Set
- Generic Sum theorem Machinery
- Sum Theorem for FVP operators

What is Monotonicity?

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- For $x^* \in X^*$ we have $x \mapsto \langle x, x^* \rangle$ a continuous linear functional on X and for each $x \in X$ we have $x^* \mapsto \langle x, x^* \rangle$ a (weak*) continuous linear functional on X^* i.e.

$$\langle \alpha_1 x_1 + \alpha_2 x_2, x^* \rangle = \alpha_1 \langle x_1, x^* \rangle + \alpha_2 \langle x_2, x^* \rangle$$

and $$\langle \alpha_1 x_1^* + \alpha_2 x_2^*, x \rangle = \alpha_1 \langle x_1^*, x \rangle + \alpha_2 \langle x_2^*, x \rangle.$$

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$$\langle \alpha_1 x_1 + \alpha_2 x_2, x^* \rangle = \alpha_1 \langle x_1, x^* \rangle + \alpha_2 \langle x_2, x^* \rangle$$
$$\text{and} \quad \langle \alpha_1 x_1^* + \alpha_2 x_2^*, x \rangle = \alpha_1 \langle x_1^*, x \rangle + \alpha_2 \langle x_2^*, x \rangle.$$

- A Banach space X is reflexive if $X^{**} = X$ i.e. the set of continuous linear forms on X^* is congruent to X.

What is a Monotone Operator?

The polar cone is given by

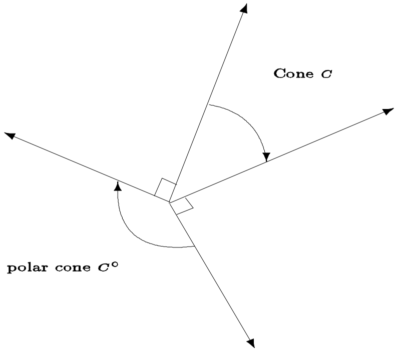

Figure:

$$C^{\circ} = \{x^* \in X^* \mid \langle x, x^* \rangle \leq 0, \quad \text{for all } x \in C\}.$$

What is a Monotone Operator?

- Note $R^n_+ = -\left(R^n_+\right)^\circ$ in finite dimensions.

Definition

An operator $T : X \to X^*$ (possibly multi-valued) is called monotone iff for all $x^* \in Tx$ and $y^* \in Ty$ we have

$$\langle x - y, x^* - y^* \rangle \geq 0.$$

What is a Monotone Operator?

- Note $R^n_+ = -\left(R^n_+\right)^\circ$ in finite dimensions.
- If from $v \leq_C w$ it follows that $Tv \leq_{-C^\circ} Tw$ we have $v - w \in C$ implies $Tw - Tv \in -C^\circ$ or

$$\langle Tw - Tv, w - v\rangle \geq 0.$$

Similarly if $w \leq_C v$ it follows that $Tw \leq_{-C^\circ} Tv$ we have $v - w \in C$ implies $Tv - Tw \in -C^\circ$ or

$$\langle Tv - Tw, v - w\rangle \geq 0, \quad \text{again.}$$

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Convex Functions

- The stock example of a "nice" monotone operator that can be multi-valued is the subgradient of a proper, extended real valued, closed convex function i.e. $\operatorname{epi}(f) := \{(x, \alpha) \mid \alpha \geq f(x)\}$ is closed, $f > -\infty$ but can take values $+\infty$ outside $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$.

Definition

Let $f : X \to \mathbf{R}_{+\infty}$ be convex. The vector $x^* \in X^*$ is a subgradient of f at $\bar{x}$ if it satisfies the following inequality for all $x \in X$

$$f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle \quad \text{(the subgradient inequality).}$$

Denote by $\partial f(\bar{x})$ the set of all subgradients of f at $\bar{x}$.

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- Clearly $\partial f(\bar{x})$ is a convex set.
- When the usual Frechet or Gateaux derivative $\nabla f(\bar{x})$ exists it is the unique subgradient.

Convex Functions is Monotone (and maximal)

- Write down the subgradient inequality twice (at x and at y)

$$f(x) - f(y) \geq \langle y^*, x - y \rangle \quad \text{and} \quad f(y) - f(x) \geq \langle x^*, y - x \rangle$$

adding implies

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- One can do this with a cycle of point x_0, $x_{1,} \ldots, x_n, x_{n+1} = x_0$ to get

$$\sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \geq 0.$$

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- Indeed ∂f needs to be "Maximal" if X is just a Banach space. That is, we can't extend it any further as a monotone operator.

Convex Functions are "mostly" single valued

- There began an industry in the 1970-1980 (i.e. Kenderov, Phelps) around the linking of "Banach Geometric" properties, such as "rotundity of the dual norm" and properties of the duality mapping

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- An Asplund space (which includes all reflexive spaces) is defined to be those Banach spaces on which, equivalently:
  - Every continuous convex function f on X is generically differentiable (generic means on a G_δ dense subset)
  - Every (maximal) monotone operator is single-values on a generic subset of its domain.

Why Maximality

- A lot of work in infinite dimension revolves around finite approximation and an easy consequence of maximality is the demi-closed property:

$$x_n^* \in T(x_n), \quad x_n^* \to_w x^*, x_n \to_s x \implies x^* \in T(x).$$

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- Sums of maximal monotone operators arise frequently and we would like to know if they are maximal?
- Outside of reflexive spaces question regarding maximality and closeness are much more difficult due to total failure of any kind of joint continuity of the duality mapping $(x, x^*) \mapsto \langle x, x^* \rangle$ with respect to any *topology* compatible with $s \times w^*$ convergence (and duality).

Maximality of Sums

As we have seen many applications require maximality of a sum of monotone operators. This issue was resolved in Reflexive spaces by Terry Rockafellar, "On the maximality of a Sum of Nonlinear Monotone Operators", Trans. Am. Math. Soc., 159, 81-99, 1970.

Theorem (Rockafellar's Sum Theorem)

Suppose that S and T are maximal monotone operators on a reflexive Banach space. Suppose that

$$\operatorname{int} \operatorname{dom}(S) \cap \operatorname{dom}(T) \neq \emptyset.$$

Then $S + T$ is maximal monotone.

Maximality is the hard part to prove, monotonicity is trivial: Suppose $x^* \in T(x)$ and $y^* \in S(x)$ then $x^* + y^* \in S(x) + T(x) = (S + T)(x)$ and for $u^* \in T(u)$ and $v^* \in S(u)$, using monotonicity of S and T,

$$\langle x^* + y^* - (u^* + v^*), x - u \rangle = \langle x^* - u^*, x - u \rangle + \langle y^* - v^*, x - u \rangle \geq 0.$$

Consequences of a Sum Theorem

- There are many consequences of a sum theorem when it holds, the domain dom T is "nearly convex" i.e. $\overline{\text{dom } T}$ is convex.

Corollary (Convex Closure, Simons)

Assume the sum theorem. Suppose T is maximal monotone then T is of type (FPV). In particular, $\text{dom}(T)$ *has a convex closure.*

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- Martínez-Legaz and Svaiter introduced the monotone polar of monotone set $T \subseteq X \times X^*$ which is denoted by

$$T^{\mu} := \{(x, x^*) \in X \times X^* \mid \langle x - y, x^* - y^* \rangle \geq 0, \, \forall \, (y, y^*) \in T\} \, .$$

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- T is of type (FPV): $\forall V \subseteq X$ with dom $T \cap V \neq \emptyset$ the following holds: if $x \in V$ and $(x, x^*) \in (T|_V)^{\mu}$ implies $x^* \in T(x)$.

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Assume the sum theorem. Suppose T is maximal monotone then T is of type (FPV). In particular, dom(T) *has a convex closure.*

Representative Functions

- Couple $X \times X^*$ with $X^* \times X$ using $\langle (z, z^*), (x^*, x) \rangle = \langle z, x^* \rangle + \langle x, z^* \rangle$ and $\|(z, z^*)\|^2 = \|z\|^2 + \|z^*\|^2$.

Definition

We call a proper convex function f is *representative* when $f(y, y^*) \geq \langle y, y^* \rangle$ for all $(y, y^*) \in X \times X^*$ and it *represents* $M_f := \{(x, x^*) \in X \times X^* \mid f(x, x^*) = \langle x, x^* \rangle\}$.

Theorem (Fitzpatrick/Penot?)

If f is representative then M_f is a monotone set.

Proof: Let $(x, x^*), (y, y^*) \in M_f$ then $\langle x - y, x^* - y \rangle \geq 0$ follows from
$\frac{1}{2}\langle x, x^* \rangle + \frac{1}{2}\langle y, y^* \rangle = \frac{1}{2} f(x, x^*) + \frac{1}{2} f(y, y^*) \geq f\left(\frac{1}{2}(x, x^*) + \frac{1}{2}(y, y^*)\right)$
$\geq \langle \frac{1}{2}(x + y), \frac{1}{2}(x^* + y^*) \rangle = \frac{1}{4}\langle x, x^* \rangle + \frac{1}{4}\langle x, y^* \rangle + \frac{1}{4}\langle y, x^* \rangle + \frac{1}{4}\langle y, y^* \rangle.$

The Fitzpatrick Function

- Define the 'transpose' operator †: $(x^*, x) \leftrightarrow (x, x^*)$ and $c_T(\cdot,\cdot) := \delta_T(\cdot,\cdot) + \langle\cdot,\cdot\rangle$, ($\delta_T$ the indicator of the graph of T). Fitzpatrick showed that

$$\mathcal{P}_T = \mathcal{F}_T^{*\dagger} \text{ and } \mathcal{F}_T = c_T^{*\dagger} \quad \text{defined on} \quad (X, X^*)$$

are representative when T is maximal. Indeed, $\mathcal{P}_T$ is the largest under the pointwise order and $\mathcal{F}_T$ the smallest. As a consequence proofs were reduced from dozens of pages to half pages.

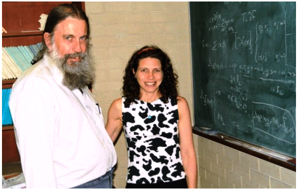

Figure: Simon Fitzpatrick and Regina Burachik

The Reflexive Case

In a reflexive space there is a beautiful characterisation of maximal monotonicity in terms of representative functions:

Denote $M_h^{\leq} := \{(x, x^*) \in X \times X^* \mid h(x, x^*) \leq \langle x, x^* \rangle\}$ then $T^\mu = M_{\mathcal{F}_T}^{\leq}$.

Theorem (Burachik and Svaiter)

Suppose X is a reflexive Banach space and $h : X \times X^* \to \mathbb{R}_{+\infty}$ be a convex lower semi-continuous function. Suppose that

$$\forall (x, x^*) \in X \times X^* \quad h(x, x^*) \geq \langle x, x^* \rangle, \quad h^{*\dagger}(x, x^*) \geq \langle x, x^* \rangle.$$

Then $T := M_h^{\leq}$ is maximal monotone and $h \in R(T)$.

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- So as to respect the basic duality relationships for conjugation on $X \times X^*$ paired with $X^* \times X$, with the later endowed with the $w^* \times s$ topology.

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- We say T is representable when there exists $f \in R(T) := \{ f \in [\mathcal{F}_T, \mathcal{P}_T] \mid f \geq \langle \cdot, \cdot \rangle \}$ with $T = M_f := M_f^{\leq}$ when f is representative.

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- It is well known that when $f \in R(T)$ then $f \in [\mathcal{F}_T, \mathcal{P}_T] = \{g \in PC(X, X^*) \mid \mathcal{F}_T \leq g \leq \mathcal{P}_T\}$, where the partial order is pointwise.

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- Recall $bR(T) := \left\{f \in [\mathcal{F}_T, \mathcal{P}_T] \mid f^{*\dagger} \geq f \geq \langle \cdot, \cdot \rangle\right\}$ are the bigger–conjugate representative functions with $T \subseteq M_f \subseteq T^\mu$.

Monotonic Closure

- One always has $T^{\mu\mu\mu} = T^{\mu}$ and if T is monotone then $T \subseteq T^{\mu}$ and $T \mapsto T^{\mu}$ is a polarity and as a consequence $A \subseteq B$ implies $A^{\mu} \supseteq B^{\mu}$, $T \subseteq T^{\mu\mu}$ and $(A \cup B)^{\mu} = A^{\mu} \cap B^{\mu}$ (for any sets $A, B \subseteq X \times X^{*}$).

Proposition (Martinez-Lagaz – Svaiter)

The following are equivalent to $T : X \rightrightarrows X^{*}$ being monotone:

1. $T \subseteq T^{\mu}$,
2. $T^{\mu\mu} \subseteq T^{\mu}$,
3. $T^{\mu\mu}$ *is monotone (with* $T \subseteq T^{\mu\mu}$*).*

We have T maximal monotone iff $T = T^{\mu}$ (or T monotone and $T \supseteq T^{\mu}$). Moreover denoting
$\mathbf{M}(T) := \{B \subseteq X \times X^{*} \mid B \text{ is maximal monotone }, T \subseteq B\}$ *then when T is monotone we have:*
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- We shall call $T^{\mu\mu}$ the monotonic closure of T.

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- Indeed if this was not the case, then we would necessarily have $(M_h)^{\mu} \subseteq M_h \subseteq (M_h)^{\mu}$ implying M_h is maximal.
- This raises the question of the existence of a representative function for the monotone set $M_h \cup \{(x, x^*)\}$? We need to include $\{(x, x^*), \langle x, x^* \rangle\}$ into the graph of the new representative function g, *via convexification*.

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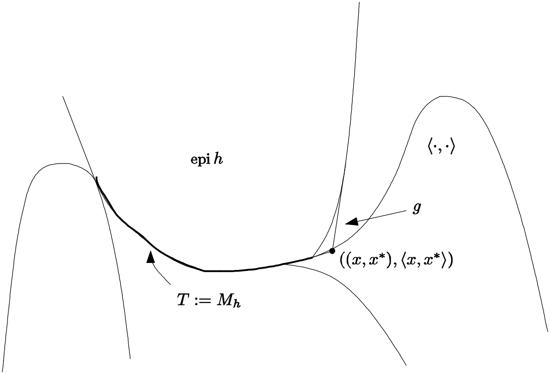
- The main idea studied here is that under the assumption that M_h for $h \in bR(T)$ is not maximal, then there exists points $(x, x^*) \in (M_h)^{\mu} \cap (M_h)^c$ (and so $\{(x, x^*)\}^{\mu} \supseteq M_h$).
- Indeed if this was not the case, then we would necessarily have $(M_h)^{\mu} \subseteq M_h \subseteq (M_h)^{\mu}$ implying M_h is maximal.
- This raises the question of the existence of a representative function for the monotone set $M_h \cup \{(x, x^*)\}$? We need to include $\{(x, x^*), \langle x, x^* \rangle\}$ into the graph of the new representative function g, *via convexification*.
- In effect we are seeking a representative function for a monotone extension of M_h to include the new point. As a consequence we quickly obtain that when $h \in bR(T)$ we have M_h monotonically closed.

Suppose $h \in bR(T)$ and that M_h is not maximal. Let $(x, x^*) \notin M_h$ and consider a convex minorant defined by

$$g := \mathrm{co}[h, \delta_{\{(x,x^*)\}} + \langle x, x^* \rangle] \leq h \tag{1}$$



- To study the problem of maximality we use the construction in (1) to define a minorising convex function $g \in b(T)$ whenever there exists a point $(x, x^*) \in (M_h)^{\mu} \cap (M_h)^c$.

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- As we clearly have $M_h \cup \{(x, x^*)\} \subseteq M_g^{\leq}$ this will only be the case if $M_g^{\leq} \subseteq (M_h \cup \{(x, x^*)\})^\mu$.

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Lemma

Suppose $h \in bR(T)$ and $(x, x^*) \in (M_h)^\mu \cap (M_h)^c$. Let g be the function constructed via (1). Then when $M_g^{\leq}$ is monotone we have

$$M_g^{\leq} \subseteq \{(x, x^*)\}^\mu . \tag{2}$$

Moreover (2) is equivalent to $M_g^{\leq} \subseteq (M_h \cup \{(x, x^*)\})^\mu$.

Theorem

(Monotone Extensions and Maximality) *Let $h \in bR(T)$ and let g be defined as in (1) using $(x, x^*) \in (M_h)^\mu \cap (M_h)^c$.*

1. *When $(z, z^*) \in \{(x, x^*)\}^\mu$ then $g(z, z^*) \geq \langle z, z^* \rangle$.*
2. *When (2) holds, then $g \in bR(M_g)$ and $g \geq \mathcal{F}_{M_g} = \mathcal{F}_{M_g^{\leq}}$.*
3. *If $\mathcal{F}_{M_h} \geq \langle \cdot, \cdot \rangle$, then $\overline{h}^{s \times w^*} \in bR(T)$.*
4. *The function*

$$f(z, z^*) := \max \{ g(z, z^*), \langle z, x^* \rangle + \langle z^*, x \rangle - \langle x, x^* \rangle \}. \tag{3}$$

   is always representative and $M_f = M_g^{\leq} \cap \{(x, x^*)\}^\mu$ is a monotone extension of $M_h \cup \{(x, x^*)\}$.
5. *If M_h is not maximal, then there cannot exist $(x, x^*) \in (M_h)^\mu \cap (M_h)^c$ such that (2) holds for g.*

As a consequence we may prove the following theorems.

Theorem

A monotone operator M is monotonically closed iff M is representable i.e. there exists $h \in bR(M)$ such that $M = M_h$. That is we always have

$$(M_h)^{\mu\mu} = M_h.$$

We say that an operator $T : X \rightrightarrows X^*$ is **completely closed** iff 1). for any bounded net $(x_\alpha, x^*_\alpha) \to (x, x^*)$ with $(x_\alpha, x^*_\alpha) \in T$ we have $(x, x^*) \in T$. 2). whenever $(x_\alpha, x^*_\alpha) \in T$ with $x_\alpha \to x \in \operatorname{dom} T$ and $\|x^*_\alpha\| \to \infty$ we have $\exists t_\alpha \to 0^+$ such that along some subnet $t_\alpha x^*_\alpha \to^{w^*} x^* \in 0^+ T(x)$.

Theorem

(Complete Closure of Representable Operators) *If have $M = M_h$ for $h \in bR(T)$ then M is completely closed.*

Sum Theorem

Then the *partial inf-convolutions* are the functions defined on $X \times Y$ by

$$\begin{aligned} F_1 \square_1 F_2 \quad &: (x, y) \mapsto \inf_{u \in X} \left[F_1(u, y) + F_2(x - u, y) \right] \\ F_1 \square_2 F_2 \quad &: (x, y) \mapsto \inf_{v \in Y} \left[F_1(x, y - v) + F_2(x, v) \right]. \end{aligned}$$

Theorem (Generic Sum Theorem tool)

Let X be a nonzero, real Banach space. Let T_1 and T_2 be maximal monotone operators from X to X^*. Suppose $F_i \in bR(T_i)$ are representative functions for T_i, for $i = 1, 2$ and

$$\bigcup_{\lambda > 0} \lambda \left[P_X \operatorname{dom} \overline{F_1} - P_X \operatorname{dom} \overline{F_2} \right] \quad \text{is a closed subspace of } X. \tag{4}$$

Then $F := \overline{F_1} \square_2 \overline{F_2}$ gives a bigger–conjugate representative function for $T_1 + T_2$ (i.e. $F \in bR(T_1 + T_2)$) for which $M_F = T_1 + T_2$. Consequently $T_1 + T_2$ is representable and hence monotonically closed i.e. $(T_1 + T_2)^{\mu\mu} = T_1 + T_2$. Note (4) is implied by $\operatorname{dom} T_1 \cap \operatorname{int} \operatorname{dom} T_2 \neq \emptyset$.

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- Let $\widetilde{T}_1 := T_1 \cap [\operatorname{dom} T_2 \times X^*] = (T_1)|_{\operatorname{dom} T_2}$.

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- Let $\widetilde{T}_1 := T_1 \cap [\operatorname{dom} T_2 \times X^*] = (T_1)|_{\operatorname{dom} T_2}$.
- Given an $x \in \operatorname{dom} T_1 \cap \operatorname{int} \operatorname{dom} T_2$ and an open convex set U with $x \in U \subseteq \operatorname{int} \operatorname{dom} T_2$, we have by the properties of polarity:

$$\begin{aligned}(x, x^*) \quad &\in \quad \left(\widetilde{T}_1\right)^{\mu} = \left(T_1|_{\operatorname{dom} T_2}\right)^{\mu} \subseteq \left(T_1|_{[U \cap \operatorname{dom} T_2]}\right)^{\mu} = \left(T_1|_U\right)^{\mu} \\ &\Longrightarrow \quad (x, x^*) \in T.\end{aligned}$$

- The FPV property for T can now be written as $\left(T|_U\right)^{\mu}|_U \subseteq T$ for any open convex neighbourhood U with $U \cap \operatorname{dom} T \neq \varnothing$.
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&\Longrightarrow \quad (x, x^*) \in T.
\end{aligned}
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- This observation motivates us to study the operator $\left(\widetilde{T}_1\right)^{\mu}|_{\operatorname{dom} T_2}$.

- The FPV property for T can now be written as $(T|_U)^\mu|_U \subseteq T$ for any open convex neighbourhood U with $U \cap \operatorname{dom} T \neq \varnothing$.
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- This observation motivates us to study the operator $\left(\widetilde{T}_1\right)^\mu|_{\operatorname{dom} T_2}$.
- Ultimately we will consider the case when $T_2 = N_C$ where $\operatorname{dom} T_1 \cap \operatorname{int} C \neq \varnothing$ and C is a closed convex set.

In the following we denote by $(T_1 + T_2)^\mu : X \rightrightarrows X^*$ the operator whose graph consists of all points monotonically related to the graph of $T_1 + T_2$.

Theorem

(Polar Sum Theorem) *Suppose T_1 and T_2 are monotone operators from X to X^*. Suppose T_2 is conic valued and that $0 \in (T_2)^\mu (x)$ for all $x \in \operatorname{dom} (T_2)^\mu$. Then*

$$\begin{aligned} (T_1 + T_2)^\mu \ (x) &= \left(\widetilde{T}_1\right)^\mu (x) + (T_2)^\mu (x) \\ \textit{for } x &\in \operatorname{dom} (T_1 + T_2)^\mu \\ &= \operatorname{dom} \left(\widetilde{T}_1\right)^\mu \cap \operatorname{dom} (T_2)^\mu , \end{aligned} \tag{5}$$

where $\widetilde{T}_1 := T_1 \cap [\operatorname{dom} T_2 \times X^*] = (T_1)\,|_{\operatorname{dom} T_2}$.

Theorem

(Sum Maximality Theorem) *Suppose T_1 and T_2 be monotone operators from X to X^* with* $\operatorname{dom} T_1 \cap \operatorname{dom} T_2 \neq \emptyset$*. Suppose T_2 is conic valued with also $0 \in T_2^{\mu}(x)$ for all $x \in \operatorname{dom} T_2^{\mu}$ and let $\widetilde{T}_1 := T_1|_{\operatorname{dom} T_2}$ and $M := \left(\widetilde{T}_1\right)^{\mu}|_{\operatorname{dom} T_2^{\mu}}$. Then*

$$M^{\mu} \subseteq (T_1 + T_2)^{\mu\mu} \subseteq M^{\mu\mu}$$

so M^{μ} is monotone, and when M is monotone then $(T_1 + T_2)^{\mu\mu}$ is maximal (i.e. $(T_1 + T_2)^{\mu}$ is pre-maximal) and hence $T_1 + T_2$ is maximal if monotonically closed.

Note that the key assumptions that establishes maximality is the monotonicity of the set $M := \left(\widetilde{T}_1\right)^{\mu}|_{\operatorname{dom} T_2}$.

When $T_2 = N_{\overline{U}}(u) := (\overline{\text{cone}}(U - u))^\circ$ we obtain identities.

Lemma

(Polarity Lemma) *Let T be maximal monotone from X to X^* and U a open convex set with* dom $T \cap U \neq \emptyset$. *Then*

$$\left(T + N_{\overline{U}}\right)^{\mu} = \left(T|_{\overline{U}}\right)^{\mu}|_{\overline{U}} + N_{\overline{U}} = \left(T|_{\overline{U}}\right)^{\mu}|_{\overline{U}} \tag{6}$$

and $$\left(T + N_{\overline{U}}\right)^{\mu\mu} = T + N_{\overline{U}} = \left(\left(T|_{\overline{U}}\right)^{\mu}|_{\overline{U}} + N_{\overline{U}}\right)^{\mu} = \left[\left(T|_{\overline{U}}\right)^{\mu}|_{\overline{U}}\right]^{\mu}. \tag{7}$$

When T is of type FPV then

$$\emptyset \neq \text{dom } T \cap U = \text{dom } \left(T|_{\overline{U}}\right)^{\mu} \cap U \tag{8}$$

and $$\overline{\text{dom } T} \cap U = \overline{\text{dom } \left(T|_{\overline{U}}\right)^{\mu}} \cap U \quad \text{a convex set.} \tag{9}$$

Suppose in addition that X admits a strictly convex re-norm. Then we have

$$\text{dom } T \cap \overline{U} = \text{dom } \left(T|_{\overline{U}}\right)^{\mu} \cap \overline{U}. \tag{10}$$

The following is an important tool for the development of the sum theorem. Essentially we need to know when $\left(T|_{\overline{U}}\right)^{\mu}|_{\overline{U}}$ is a monotone set. The implication (11) suffices as it contains this set in another monotone set.

Proposition

Suppose $T : X \rightrightarrows X^*$ is maximal monotone of type FPV and $U \subseteq X$ is an open convex set such that $U \cap \operatorname{dom} T \neq \varnothing$. Then we have

$$(u, u^*) \in \left(T|_{\overline{U}}\right)^{\mu}|_{\overline{U}} \quad \Longrightarrow \quad u^* + x^* \in \overline{T(u) + N_{\overline{U}}(u)}, \text{ for all } x^* \in N_{\overline{U}}(u). \tag{11}$$

when one of the following additional assumptions hold:

1. *The convex set $\overline{U}$ is strictly convex (i.e. for any $(u, u^*), (v, v^*) \in \operatorname{bd} \overline{U}$ we have $\lambda (u, u^*) + (1-\lambda)(v, v^*) \in U$ for some $\lambda \in (0, 1)$).*
2. *The space X admits a strictly convex re-norm.*

In particular $\left(T|_{\overline{U}}\right)^{\mu}|_{\overline{U}}(u) \subseteq \overline{T(u) + N_{\overline{U}}(u)}$ for all u.

- In view of Theorem 11 we have the following result which appears to not be a consequence of the known sum theorems for operators of type FPV where additional structural assumptions are made on dom T.

Theorem (Normal Cone Sums)

Suppose X is a real Banach space that admits a strictly convex re-norm, T is maximal monotone of type FPV and C is closed and convex with

$$\text{dom}\, T \cap \operatorname{int} C \neq \emptyset. \tag{12}$$

Then $T + N_C$ is maximal monotone.

- In view of Theorem 11 we have the following result which appears to not be a consequence of the known sum theorems for operators of type FPV where additional structural assumptions are made on dom T.
- We note that the existence of a strictly convex re-norm is a very weak assumption and such space contain reflexive Banach space, separable Banach spaces, WCG spaces, duals of Asplund spaces due to the duality between Gateau differentiability and strict convexity of the dual norm.

Theorem (Normal Cone Sums)

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Then $T + N_C$ is maximal monotone.

We may now obtain a characterisation of the FPV property.

Corollary

Suppose X is a real Banach space with a strictly convex re-norm, T is maximal monotone and $C \subseteq X$ is closed, convex with (12) holding. Then $T + N_C$ is maximal monotone iff T is of type FPV.

It is still possible that this basic sum theorem fails for some pathological operators. This sum theorem issue can now be resolved by the resolution of the following question, which to the authors knowledge still remain open.

Questions: Does there exists a maximal monotone operator on a real Banach space X which is not of type FPV?

Does there exists FPV maximal monotone operators that fail to admit a sum theorem with a normal cone of a (non)strictly convex set outside the strictly convex re-normable spaces?

We finish by observing the following.

Theorem

If T is maximal monotone on a real Banach space X that admits a strictly convex re-norm. Then consider the following are all equivalent:

1. *T is of type FPV i.e. for all open convex sets U we have* $(y, y^*) \in (T|_U)^{\mu}|_U \implies (y, y^*) \in T.$
2. *Whenever $U \subseteq X$ is an open, convex set such that $U \cap \operatorname{dom} T \neq \emptyset$, then we have*

$$(y, y^*) \in \left(T|_{\overline{U}}\right)^{\mu}|_{\overline{U}} \implies (y, y^* + x^*) \in T + N_{\overline{U}}, \text{ for any } x^* \in N_{\overline{U}}(y)\,. \tag{13}$$

3. *Whenever C is closed convex with $\operatorname{dom} T \cap \operatorname{int} C \neq \emptyset$. Then we have*

$$(T|_C)^{\mu}|_C \subseteq T + N_C.$$

4. *When C is a closed, convex set with $\operatorname{dom} T \cap \operatorname{int} C \neq \emptyset$ then $T + N_C$ is maximal monotone.*

Thank You!

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