Union Averaged Operators with Applications to Proximal Algorithms for Min-Convex Functions

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THE UNIVERSITY OF NEWCASTLE AUSTRALIA

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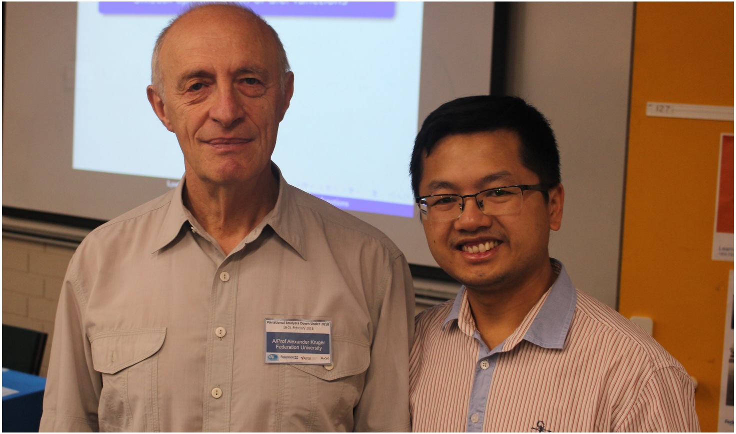
Joint work with Matthew K. Tam (Universität Göttingen, Germany)

Happy Birthday, Alex!



Ballarat, February 2018

Outline

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Projectors and reflectors

Throughout this talk,

$$X := \mathbb{R}^d.$$

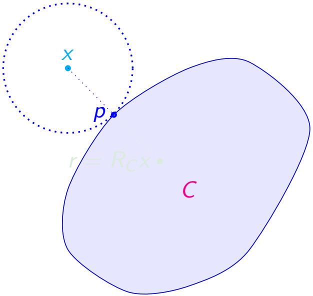
Given a nonempty closed subset C of X, the projector onto C is defined by

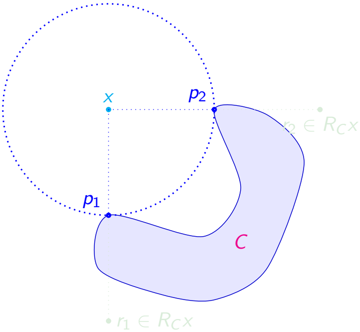
$$P_C\colon X \rightrightarrows C\colon x \mapsto P_C x := \operatorname{argmin}_{c\in C} \|x - c\|$$

and reflector across C is

$$R_C := 2P_C - \mathrm{Id},$$

where Id is the identity operator on X.





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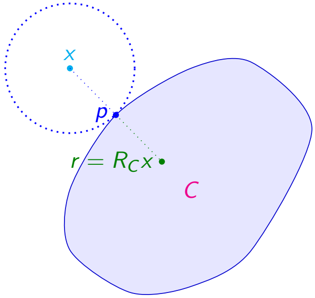

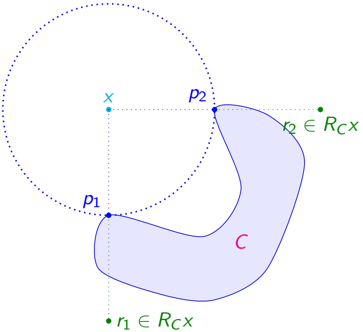

Alternating projections (AP) and Douglas–Rachford (DR) algorithm

Let A and B are closed sets with $A \cap B \neq \varnothing$. The feasibility problem is to

$$\text{find } x \in A \cap B.$$

- AP operator: $P_B P_A$.

- DR operator: $T_{A,B} := \frac{1}{2}(\mathrm{Id} + R_B R_A)$ (a.k.a. reflect-reflect-average).

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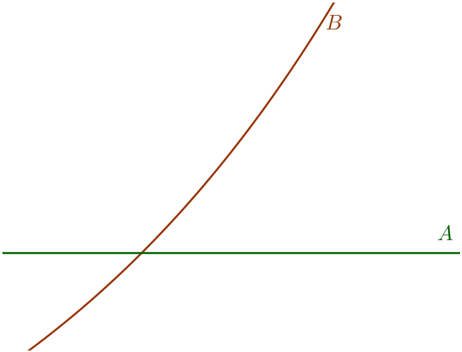

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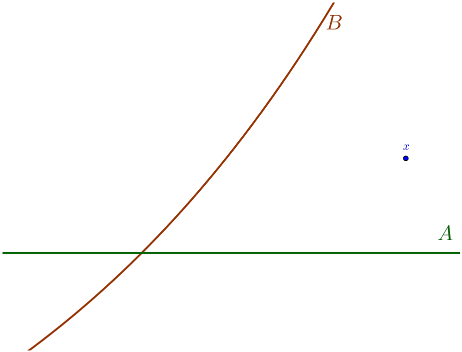

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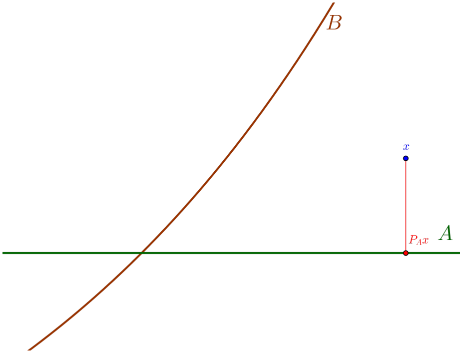

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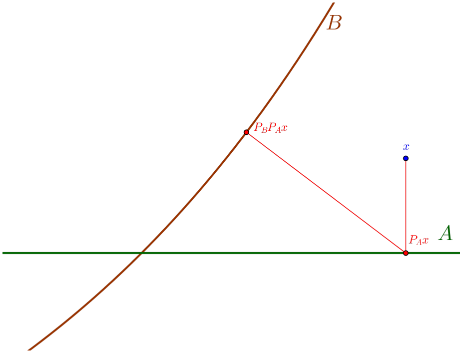

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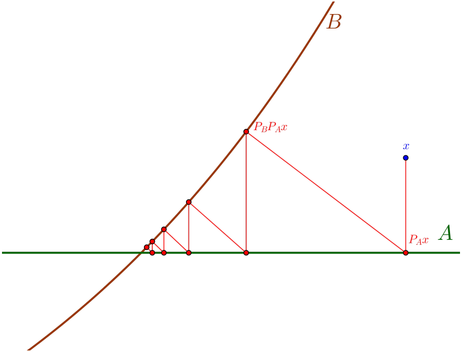

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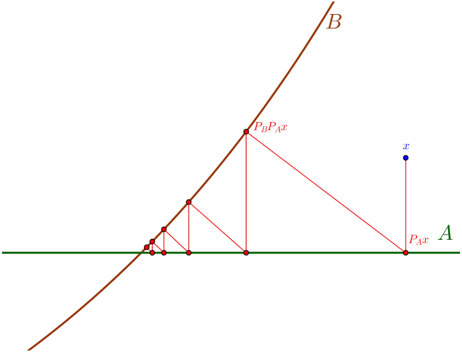

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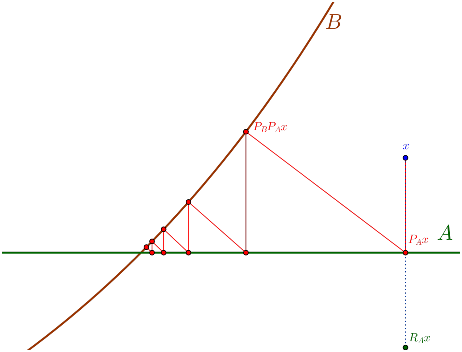

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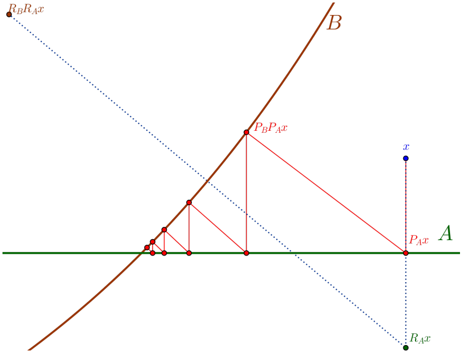

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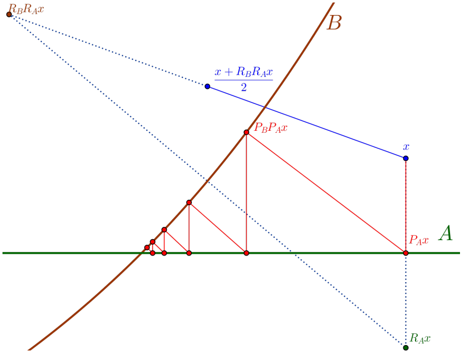

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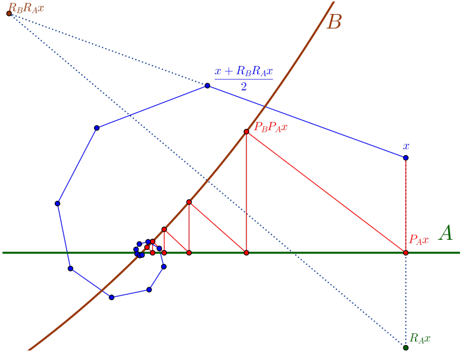

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Convergence of AP and DR algorithms

When A and B are convex,

- the AP algorithm is globally convergent to a point in the intersection (Bregman, 1965);
- the DR algorithm is globally convergent to a fixed point (Lions–Mercier, 1979).

When $A = \cup_{i \in I} A_i$ and $B = \cup_{j \in J} B_j$ are finite unions of closed convex sets,

- the DR algorithm is locally convergent around strong fixed points (Bauschke–Noll, 2014);
- is the AP algorithm locally convergent?

For a set-valued operator $T\colon X \rightrightarrows X$, the fixed point set is $\operatorname{Fix} T := \{x : x \in T(x)\}$, and the strong fixed point set is $\mathbf{Fix}\, T := \{x : \{x\} = T(x)\}$. Both notions coincide for single-valued operators.

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Averaged operators

Recall that a single-valued operator $T\colon X \to X$ is nonexpansive if

$$\forall x, y \in X, \quad \|Tx - Ty\| \leq \|x - y\|,$$

and α-averaged if $\alpha \in (0, 1)$ and

$$\forall x, y \in X, \quad \|Tx - Ty\|^2 + \frac{1-\alpha}{\alpha}\|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \leq \|x - y\|^2.$$

- T is α-averaged if and only if $T = (1-\alpha)\,\mathrm{Id} + \alpha R$ for some nonexpansive operator $R\colon X \to X$.
- The classes of nonexpansive and averaged operators are both closed under taking convex combination and under compositions.

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Union averaged operators

Definition

A set-valued operator $T\colon X \rightrightarrows X$ is said to be union α-averaged (resp. union nonexpansive) if T can be expressed in the form

$$\forall x \in X, \quad T(x) = \{T_i(x) : i \in \varphi(x)\},$$

where

- I is a finite index set,
- $\{T_i\}_{i \in I}$ is a collection of α-averaged (resp. nonexpansive) operators,
- $\varphi\colon X \rightrightarrows I$, called an active selector, is nonempty-valued and outer semicontinuous (osc):

$$\varphi(x) \supseteq \operatorname{Limsup}_{y \to x} \varphi(y) := \{i \in I : \exists\, (x_n, i_n) \to (x, i) \text{ with } i_n \in \varphi(x_n)\}.$$

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Unions, combinations and compositions

Proposition

Let $J := \{1, \ldots, m\}$ and let $T_j \colon X \rightrightarrows X$ be union α_j-averaged (resp. union nonexpansive) for each $j \in J$. Then

1. $T \colon X \rightrightarrows X$ *defined by* $x \mapsto T(x) := \cup_{j \in J} T_j(x)$ *is union α-averaged with* $\alpha := \max_{j \in J} \alpha_j$ *(resp. union nonexpansive).*
2. $\sum_{j \in J} \omega_j T_j$ *is union α-averaged with* $\alpha := \sum_{j \in J} \omega_j \alpha_j$ *(resp. union nonexpansive) whenever* $(\omega_j)_{j \in J} \subseteq \mathbb{R}_{++}$ *with* $\sum_{j \in J} \omega_j = 1$.
3. $T_m \circ \cdots \circ T_2 \circ T_1$ *is union α-averaged with*

$$\alpha := \left(1 + \left(\sum_{j \in J} \frac{\alpha_j}{1 - \alpha_j}\right)^{-1}\right)^{-1}$$

(resp. union nonexpansive).

Min-convex functions

Definition

A function $f\colon X \to (-\infty, +\infty]$ is said to be min-convex if

$$\forall x \in X, \quad f(x) := \min_{i \in I} f_i(x),$$

where I is a finite index set and the $f_i\colon X \to (-\infty, +\infty]$ are proper lsc convex functions.

If $\{C_i\}_{i \in I}$ is a finite collection of nonempty closed convex subsets of X, then

$$\iota_C = \min_{i \in I} \iota_{C_i}$$

is a min-convex function.

The indicator function ι_C of C is defined by $\iota_C(x) := 0$ if $x \in C$ and $\iota_C(x) := +\infty$ otherwise.

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Proximity operator of min-convex functions

Given $f\colon X \to (-\infty, +\infty]$ and $\gamma > 0$, the Moreau envelope of f is the function ${}^{\gamma}f\colon X \to (-\infty, +\infty]$ given by

$$ {}^{\gamma}f(x) := \inf_{y \in X} \left(f(y) + \frac{1}{2\gamma}\|x - y\|^2 \right) $$

and the proximity operator of f is the mapping $\operatorname{prox}_{\gamma f}\colon X \rightrightarrows X$ given by

$$ \operatorname{prox}_{\gamma f}(x) = \left\{ y \in X : f(y) + \frac{1}{2\gamma}\|x - y\|^2 = {}^{\gamma}f(x) \right\}. $$

Proposition

Let $f = \min_{i \in I} f_i\colon X \to (-\infty, +\infty]$ be min-convex, and $\gamma > 0$. Then

1. *Every fixed point of $\operatorname{prox}_{\gamma f}$ is a local minimum of f.*
2. $\operatorname{prox}_{\gamma f}$ *is union 1/2-averaged. In particular,*

$$ \operatorname{prox}_{\gamma f}(x) = \{ \operatorname{prox}_{\gamma f_i}(x) : i \in \varphi(x) \} $$

with $\varphi\colon X \rightrightarrows I$ given by $\varphi(x) = \{ i \in I : {}^{\gamma}f(x) = {}^{\gamma}f_i(x) \}$.

Projection operators on union convex sets

Corollary

Let $A = \cup_{i\in I} A_i$ and $B = \cup_{j\in J} B_j$ be finite unions of nonempty closed convex sets in X. Then

1. *The projector P_A is union $1/2$-averaged with*

$$P_A(x) = \{P_{A_i}(x) : i \in I,\ d(x, A_i) = d(x, A)\}$$

   and the reflector $R_A := 2P_A - \mathrm{Id}$ is union nonexpansive.

2. *The DR operator $T_{A,B}$ is union $1/2$-averaged.*

$d(x, C) := \inf_{c\in C} \|x - c\|$ is the distance from x to C.

Outline

Krasnosel'skiĭ–Mann iterations with admissible control

Definition (Admissible sequences)

A sequence $(i_n)_{n\in\mathbb{N}} \subseteq I$ is admissible in $I^* \subseteq I$ if every element of I^* appears infinitely often in $(i_n)_{n\in\mathbb{N}}$.

Theorem

Let $\{T_i\}_{i\in I}$ be a finite collection of nonexpansive operators on X with a common fixed point. Given $x_0 \in X$, define $(x_n)_{n\in\mathbb{N}}$ according to

$$\forall n \in \mathbb{N}, \quad x_{n+1} := (1-\lambda_n)x_n + \lambda_n T_{i_n}(x_n),$$

where $(i_n)_{n\in\mathbb{N}}$ is admissible in I, and $(\lambda_n)_{n\in\mathbb{N}}$ is in $(0,1]$ with $\liminf_{n\to\infty} \lambda_n(1-\lambda_n) > 0$. Then $x_n \to \overline{x} \in \cap_{i\in I} \operatorname{Fix} T_i$.

- The T_i are α_i-averaged
  $\longrightarrow \lambda_n \in (0, 1/\alpha_{i_n}]$ and $\liminf_{n\to\infty} \lambda_n(1-\alpha_{i_n}\lambda_n) > 0$.

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Local convergence of union nonexpansive iterations

Let $T\colon X \rightrightarrows X$ be a union nonexpansive operator with representation

$$T(x) = \{T_i(x) : i \in \varphi(x)\}.$$

The radius of attraction of T at a point $x^* \in X$ is defined as

$$r(x^*; T) := \sup\{\delta > 0 : \forall x \in \mathbb{B}(x^*;\delta),\ \varphi(x) \subseteq \varphi(x^*)\} \in (0, +\infty].$$

Theorem

Let $x^* \in \mathbf{Fix}\, T$*, set* $r := r(x^*; T)$*, and consider a sequence* $(x_n)_{n\in\mathbb{N}}$ *with* $x_0 \in \operatorname{int} \mathbb{B}(x^*; r)$ *satisfying*

$$\forall n \in \mathbb{N}, \quad x_{n+1} \in (1-\lambda_n)x_n + \lambda_n T(x_n),$$

where $(\lambda_n)_{n\in\mathbb{N}} \subseteq (0,1]$ *with* $\liminf_{n\to\infty} \lambda_n(1-\lambda_n) > 0$*. Then*

$$x_n \to \overline{x} \in \text{Fix}\, T \cap \mathbb{B}(x^*; r).$$

- T is union α-averaged: $\lambda_n \in (0, 1/\alpha]$ & $\liminf_{n\to\infty} \lambda_n(1/\alpha - \lambda_n) > 0$.
- $\varphi(x^*) = I \implies r(x^*; T) = +\infty \longrightarrow$ global convergence.

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*Let $x^* \in$ **Fix** T, set $r := r(x^*; T)$, and consider a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_0 \in \operatorname{int} \mathbb{B}(x^*; r)$ satisfying*

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Outline

Projection algorithms

Theorem

Let $J := \{1, \dots, m\}$ and let $\{C_j\}_{j\in J}$ be a collection of union convex sets. Given $x_0 \in X$, define $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N}$ in one of the following:

1. *(method of cyclic projections)* $T = P_{C_m} \circ \cdots \circ P_{C_2} \circ P_{C_1}$.
2. *(cyclic DR method)* $T = T_{C_m,C_1} \circ \cdots \circ T_{C_2,C_3} \circ T_{C_1,C_2}$.
3. *(cyclically anchored DR method)* $T = T_{C_1,C_m} \circ \cdots \circ T_{C_1,C_3} \circ T_{C_1,C_2}$.

Then $\cap_{j\in J} C_j \subseteq \textbf{Fix}\, T$. Moreover, if $x^* \in \textbf{Fix}\, T$, then

$$\exists r > 0,\ \forall x_0 \in \operatorname{int} \mathbb{B}(x^*; r), \quad x_n \to \overline{x} \in \operatorname{Fix} T \cap \mathbb{B}(x^*; r).$$

Cyclically anchored DR method

Theorem

Let $J := \{1, \dots, m\}$ *and let* $\{C_j\}_{j \in J}$ *be a collection of union convex sets with* $x^* \in \bigcap_{j \in J \setminus \{1\}} \mathbf{Fix}\, T_{C_1, C_j}$. *Given* $x_0 \in X$, *define* $(x_n)_{n \in \mathbb{N}}$ *according to*

$$\forall n \in \mathbb{N}, \quad x_{n+1} \in T_{C_1, C_{i_n}}(x_n) \quad \textit{where} \quad i_n = (n \bmod (m-1)) + 2.$$

Then $\exists r > 0$, $\forall x_0 \in \operatorname{int} \mathbb{B}(x^*; r)$, $x_n \to \overline{x} \in \bigcap_{j \in J \setminus \{1\}} \operatorname{Fix} T_{C_1, C_j}$.
Moreover, if the set C_1 *is convex, then* $P_{C_1}(\overline{x}) \in \bigcap_{j \in J} C_j$.

- $m = 2$: Result by Bauschke–Noll (2014).
- Apply to $\{X, C_1, \dots, C_m\}$: Result for the method of cyclic projections.

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Then $\exists r > 0$, $\forall x_0 \in \operatorname{int} \mathbb{B}(x^*; r)$, $x_n \to \overline{x} \in \bigcap_{j \in J \setminus \{1\}} \operatorname{Fix} T_{C_1, C_j}$. Moreover, if the set C_1 is convex, then $P_{C_1}(\overline{x}) \in \bigcap_{j \in J} C_j$.

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Proximal algorithms

Let $f: X \to \mathbb{R}$ be convex, $g := \min_{i \in I} g_i : X \to (-\infty, +\infty]$ be min-convex, and $\gamma > 0$.

- Proximal point algorithm (PPA): $T_{\mathrm{PPA}} := \mathrm{prox}_{\gamma g}$.
- Forward–backward splitting (FBS): $T_{\mathrm{FB}} := \mathrm{prox}_{\gamma g}(\mathrm{Id} - \gamma \nabla f)$.
- Douglas–Rachford splitting (DRS):

$$T_{\mathrm{DR}} := \frac{1}{2}\left(\mathrm{Id} + (2\,\mathrm{prox}_{\gamma g} - \mathrm{Id}) \circ (2\,\mathrm{prox}_{\gamma f} - \mathrm{Id})\right).$$

Theorem

- *The PPA is locally convergent to a local minimum of g.*
- *The DRS is locally convergent to a fixed point $\overline{x}$ such that $\mathrm{prox}_{\gamma f}(\overline{x})$ is a local minimum of $f + g$.*
- *If f has Lipschitz continuous gradient, then FBS is locally convergent to a local minimum of $f + g$.*

Some key references

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THANK YOU VERY MUCH!