ANCIENT TRANSVERSALITY

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Transversality is about
separation and intersection of possibly convex sets. Separation is closely related to extensions of linear mappings.
So this talk will be about
separation; intersection of balls; extensions of linear operators.

The geometric form of the Hahn(1927)-Banach(1929) theorem, also known as the **separation theorem**, has numerous uses in convex geometry, optimization theory, functional analysis, and elsewhere. The separation theorem is equivalent to the extension form of the **Hahn-Banach theorem**, of which we state the simplest version now.

Theorem

If $f : M \to \mathbb{R}$ is a continuous linear functional defined on a linear subspace M of a normed space X, then there exists a linear extension $F : X \to \mathbb{R}$ of f to the whole space X, with $\|F\| = \|f\|$.

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We will concentrate on the former, and examine the proof now.

If X is only one dimension bigger than M, we can define a linear extension of f by specifying its value at a single point $v \in X \setminus M$. How do we ensure that the extension F has the same norm?
Any $x \in X$ can be written as $x = m + \lambda v$ for some $m \in M$, $\lambda \in \mathbb{R}$. So we need to check that

$$|F(m + \lambda v)| \leqslant \|f\| \|m + \lambda v\|$$

always holds. We can assume $\|f\| = 1$. A little rearranging shows that we need to have

$$f(m) - \|v - m\| \leqslant F(v) \leqslant f(m) + \|v - m\|$$

for every $m \in M$. Is this possible?

The triangle inequality implies

$$f(m_1) - \|v - m_1\| \leqslant f(m_2) + \|v - m_2\|$$

for all $m_1, m_2 \in M$ and so

$$\sup_{m \in M} f(m) - \|v - m\| \leqslant \inf_{m \in M} f(m) + \|v - m\|.$$

Thus we can choose $F(v)$ to be any number between

$$\sup_{m \in M} f(m) - \|v - m\|$$

and

$$\inf_{m \in M} f(m) + \|v - m\|.$$

The question now is: what properties of $\mathbb{R}$ make this proof work? What other target spaces can we replace $\mathbb{R}$ with, and still get a useful result? On what essential property of the real numbers does the usual proof of the extensibility of linear functionals rest?

Gleb Akilov was the first to attack this problem, in 1947. Realising the role played by order in this proof, he worked with normed vector lattices which had recently been introduced by Kantorovitch. Amongst other things, he proved the following.

Theorem

Let Y be a Banach lattice in which every set which is bounded above possesses an exact supremum and the unit ball of Y has a maximal element. Then Y may replace $\mathbb{R}$ in the statement of the Hahn-Banach Theorem.

However the property of $F(v)$ which we required may also be written in the following way:

$$F(v) \in [f(m) - \|v - m\|, f(m) + \|v - m\|]$$

for every $m \in M$.

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for every $m \in M$. Now an interval $[a, b]$ in $\mathbb{R}$ is also a closed ball (with centre $\frac{1}{2}(a + b)$ and radius $\frac{1}{2}|a - b|$).
Leopoldo Nachbin (1949) abstracted out this property of Banach space, now known as the **binary intersection property**: every collection of closed balls, any two members of which intersect, has nonempty intersection.

Theorem

A real normed space has the extension property if and only if its closed balls have the binary intersection property.

Several authors (Dwight Goodner, Morisuke Hasumi, John Kelley) built on all this, to show that a Banach space has the extension property if and only if it is equivalent to a space $C(K)$ of all real-valued continuous functions on some totally disconnected compact Hausdorff space K. Total disconnectedness is a strange property; examples include finite spaces, and the Stone-Čech compactifications of discrete sets.

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J.A.R. Holbrook (1975) and O. Hustad (1974) independently proved that the complex (and the real) numbers have the following intersection property.

A finite collection $B(x_i, r_i)$ of closed balls has non-empty mutual intersection $\bigcap_i B(x_i, r_i)$ if and only if

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A finite collection $B(x_i, r_i)$ of closed balls has non-empty mutual intersection $\bigcap_i B(x_i, r_i)$ if and only if
whenever λ_i are scalars with $\sum_i \lambda_i = 0$, then

$$\left|\sum_i \lambda_i x_i\right| \leqslant \sum_i |\lambda_i| r_i.$$

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Using this formally weaker intersection property instead of the binary intersection property then gives a natural proof of the Hahn-Banach Theorem for both scalar fields.

Carrying on, a collection $B(x_i, r_i)$ of closed balls in a Banach space X is declared to have the *weak intersection property* if and only if whenever λ_i are scalars with $\sum_i \lambda_i = 0$, then

$$\left\| \sum_i \lambda_i x_i \right\| \leqslant \sum_i |\lambda_i| r_i.$$

With the same argument, it follows that a (real or complex) Banach space has the extension property if and only if every family of balls with the weak intersection property actually has non-empty mutual intersection.

Aronszajn and Panitchpakdi (1956) also considered the properties defined by collections of intersecting balls with limited cardinality. This theme was continued by Joram Lindenstrauss (1964), who defined a sequence of weaker intersection properties:
A Banach space has the *n*.2.i.p. if, for every *n* open balls $B_1, B_2, \ldots, B_n$ which intersect pairwise, the mutual intersection $\bigcap_{i=1}^{n} B_i$ is non-empty.

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Theorem

For any real Banach space Y, the following are equivalent.
i) Y has the 4.2.i.p,
ii) Y has the n.2.i.p for every n,
iii) Y^{**} has the Hahn-Banach extension property,
iv) Y^* is isometric to $L_1(\mu)$ for some measure μ,
v) for every Banach space X, subspace M, compact operator $k : M \to Y$, there is a compact extension $K : X \to Y$ with almost the same norm.

Now we introduce a subspace into the mix of intersecting balls: a subspace J is said to have the n-ball property in X if, whenever $B_1, \ldots, B_n$ are open balls in X, with $\bigcap_{i=1}^n B_i$ non-empty and $J \cap B_i$ non-empty for each i, then we also have $J \cap \bigcap_{i=1}^n B_i$ non-empty.

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It is well known now that the 3-ball property implies the n-ball property for all n, and this happens if and only if J^0 is an L-summand in X^*; such subspaces are called M-ideals. Examples of M-ideals include any ideal in a C^*-algebra; many ideals in uniform algebras; and the compact operators in $B(\ell_p)$, for $1 < p < \infty$.

These concepts were introduced by Alfsen and Effros (1973).

Lima (1977) gave a more elegant presentation of the basic theory of M-ideals, by pushed the ideas of Hustad further:

Theorem

Let X be a Banach space, J a closed subspace, and fix open balls $B(a_1, r_1) \ldots, B(a_n, r_n)$ in X. Then the following are equivalent.

i. $J \cap \bigcap_{i=1}^{n} B(a_i, r_i)$ *is non-empty,*
ii. *whenever* $f_1, \ldots, f_n \in X^*$ *satisfy* $\sum_i f_i \in J^0$ *then* $|\sum_i f_i(a_i)| \leqslant \sum_i r_i \|f_i\|$.

In other words, mutual intersections of a family of balls and a subspace is described by an inequality involving functionals whose sum annihilates the subspace.
This is proved by embedding the family into a product space, and trying to separate it from the diagonal.

The literature on M-ideals is now vast, with applications to operator algebras, harmonic analysis, approximation theory and other fields.
Enough . . .


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