On differential variational inequalities associated with solution schemes for solving maximally monotone inclusion problems

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Contents

Operator inclusions problem

Let $\mathcal{H}, \mathcal{G}$ be Hilbert spaces, $A: \mathcal{H} \to \mathcal{H}$, $B: \mathcal{G} \to \mathcal{G}$ be maximally monotone operators and $L: \mathcal{H} \to \mathcal{G}$ be a linear, bounded continuous operator. We are interested in finding a point $u \in \mathcal{H}$ which solves the following inclusion problem

$$0 \in Au + L^* BLu. \tag{P}$$

The dual inclusion problem to (5) is to find $v^* \in \mathcal{G}$ such that

$$0 \in -LA^{-1}(-Lv^*) + B^{-1}v^*. \tag{D}$$

A point $u \in \mathcal{H}$ solves (5) if and only if $v^* \in \mathcal{G}$ solves (D) and $(u, v^*) \in Z$, where

$$Z := \{(u, v^*) \in \mathcal{H} \times \mathcal{G} \mid -L^* v^* \in Au \quad \text{and} \quad Lu \in B^{-1}v^*\}.$$

Z is a closed convex set. We assume that Z is nonempty.
The aim is to find a point from Z.

The approach

The idea of Eckstein and Svaiter is to construct halfspaces satisfying

$$Z \subset H_{\varphi} := \{(u, v^*) \in \mathcal{H} \times \mathcal{H} \mid \varphi(u, v^*) \leq 0\}$$

(in their original formulation $L = Id$), with

$$\varphi(u, v^*) := \langle u - b \mid b^* - v^* \rangle + \langle u - a \mid a^* + v^* \rangle, \quad (a, a^*) \in \text{gph}A,\ (b, b^*) \in \text{gph}B.$$

This idea has been continued by Zhang and Cheng, Alatoibi and Combettes and Shahzad.

J. Eckstein and B. F. Svaiter. "A family of projective splitting methods for the sum of two maximal monotone operators". In: *Mathematical Programming* 111.1 (2008), pp. 173–199. ISSN: 1436-4646. DOI: 10.1007/s10107-006-0070-8. URL: http://dx.doi.org/10.1007/s10107-006-0070-8

Hui Zhang and Lizhi Cheng. "Projective splitting methods for sums of maximal monotone operators with applications". In: *Journal of Mathematical Analysis and Applications* 406.1 (2013), pp. 323 –334. ISSN: 0022-247X. DOI: https://doi.org/10.1016/j.jmaa.2013.04.072

Relation to convex optimization problems

$A = \partial f$, $B = \partial g$ - subdifferentials of convex functions f and g,
$f : \mathcal{H} \to]-\infty, +\infty]$ $g : \mathcal{G} \to]-\infty, +\infty]$, proper, l.s.c.

under some constraint qualification our problem corresponds to the minimization

$$\text{minimize}_{x \in \mathcal{H}} \ f(x) + g(Lx)$$

the Fenchel-Rockafellar dual problem

$$\text{minimize}_{v^* \in \mathcal{G}} \ f^*(-L^* v^*) + g^*(v^*)$$

and the associated Kuhn–Tucker set is the set Z coincides with (3)

$$Z = \{(x, v^*) \in \mathcal{H} \times \mathcal{G} \mid -L^* v^* \in \partial f(x) \text{ and } Lx \in \partial g^*(v^*)\}$$

the set Z is a natural extension of the Kuhn-Tucker set

Inspiration - Eckstein and Svaiter, 2007 - decomposable separator

1. original problem: $0 \in Ax + Bx$ (no L and $\mathcal{H} = \mathcal{G}$)
2. extended solution set
$$S_e(A, B) = \{(x, v^*) \in \mathcal{H} \times \mathcal{H} \mid -v^* \in Ax \;\text{ and }\; v^* \in Bx\}$$
3. this is the Kuhn-Tucker set Z
4. Fact 1: $x \in (A+B)^{-1}(0) \Leftrightarrow \exists\, v^* \in \mathcal{H}$ s.t. $(x, v^*) \in S_e(A, B)$
$$\text{proof: } 0 \in Ax + Bx \;\equiv\; \exists\; v^* \in \mathcal{H} \quad -v^* \in Ax \;\text{ and }\; v^* \in Bx$$
5. Fact 2: $A, B : \mathcal{H} \rightrightarrows \mathcal{H}$, then $S_e(A, B)$ is closed and convex
6. Let $(b, b^*) \in \text{Gph}B$ and $(a, a^*) \in \text{Gph}A$ and let $\varphi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$
$$\varphi(x, v^*) := \langle x - b, b^* - v^* \rangle + \langle x - a, a^* + v^* \rangle$$
7. Fact 3. Given $(b, b^*) \in \text{Gph}B$ and $(a, a^*) \in \text{Gph}A$. We have
   1. $S_e(A, B) \subset \{(x, v^*) \in \mathcal{H} \times \mathcal{H} \mid \varphi(x, v^*) \leq 0\}$
   2. additionally: φ is both continuous and affine,
$$\nabla \varphi = 0 \;\Leftrightarrow\; (b, b^*) \in S_e(A, B),\; b = a,\, a^* = -b^*$$

Inspiration - Eckstein and Svaiter, 2007 - the resulting algorithm

for $k = 0, 1, \dots$ **do**, start with arbitrary $p_0 = (x_0, v_0^*) \in \mathcal{H} \times \mathcal{H}$

  Choose $(b_k, b_k^*) \in \mathrm{Gph}B$, $(a_k, a_k^*) \in \mathrm{Gph}A$

  Define

$$\varphi_k(x, v^*) := \langle x - b_k, b_k^* - v^* \rangle + \langle x - a_k, a_k^* + v^* \rangle$$

  Compute $\bar{p}_k = (\bar{x}_k, \bar{v}_k^*)$ **to be the projection of** $p_k = (x_k, v_k^*)$ **onto**

$$H_k := \{(x, v^*) \in \mathcal{H} \times \mathcal{H} \mid \varphi_k(x, v^*) \leq 0\}$$

  Choose a relaxation parameter $\rho_k \in (0, 2)$ **and let**

$$p_{k+1} := p_k + \rho_k(\bar{p}_k - p_k)$$

end for

Rezolvent of subdifferential $J_{\lambda \partial f}$

1. $\lambda > 0$, $z = J_{\lambda \partial f}(x) = (I + \lambda \partial f)^{-1}(x)$, i.e. $x \in z + \lambda \partial f(z)$
2. can be rewritten: $0 \in \partial_z (f(z) + \frac{1}{2}\lambda \|z - x\|_2^2)$
3. which is the same as: $z = \mathbf{argmin}_u (f(u) + \frac{1}{2}\lambda \|u - x\|_2^2) = \mathbf{Prox}_{\lambda f}(x) = J_{\lambda \partial f}(x)$

Successive Fejér approximations iterative scheme

Let $\{H_n\}_{n\in\mathbb{N}} \subset \mathcal{H} \times \mathcal{G}$, be a sequence of convex closed sets such that $Z \subset H_n,\ n \in \mathbb{N}$. The projections of any $x \in \mathcal{H} \times \mathcal{G}$ onto H_n are uniquely defined.

Algorithm 1 Generic primal-dual Fejér Approximation Iterative Scheme

1: Choose an initial point $x_0 \in \mathcal{H} \times \mathcal{G}$
2: Choose a sequence of parameters $\{\lambda_n\}_{n\geq 0} \in (0, 2)$
3: **for** $n = 0, 1 \ldots$ **do**
4: $\quad x_{n+1} = x_n + \lambda_n(P_{H_n}(x_n) - x_n)$
5: **end for**

Convergence result

Theorem

For any sequence generated by Iterative Scheme 1 the following hold:

1. $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}\times\mathcal{G}$ *is Fejér monotone with respect to the set* Z, *i.e*

$$\forall_{n\in\mathbb{N}}\ \forall_{z\in Z}\ \|x_{n+1}-z\| \leq \|x_n - z\|,$$

2. $\sum_{n=0}^{+\infty} \lambda_n(2-\lambda_n)\|P_{H_n}(x_n)-x_n\|^2 < +\infty,$
3. *if*

$$\forall x\in\mathcal{H}\times\mathcal{G}\ \forall\{k_n\}_{n\in\mathbb{N}}\subset\mathbb{N}\quad x_{k_n}\rightharpoonup x \implies x\in Z,$$

then $\{x_n\}_{n\in\mathbb{N}}$ *converges weakly to a point in* Z.

$$\begin{aligned} H_{a_n,b_n^*} &:= \left\{x\in\mathcal{H}\times\mathcal{G} \mid \left\langle x \mid s^*_{a_n,b_n^*}\right\rangle \leq \eta_{a_n,b_n^*}\right\}, \\ s^*_{a_n,b_n^*} &:= (a_n^* + L^* b_n^*, b_n - La_n), \eta_{a_n,b_n^*} := \langle a_n \mid a_n^*\rangle + \langle b_n \mid b_n^*\rangle, \end{aligned} \tag{1}$$

with

$$\begin{aligned} a_n &:= J_{\gamma_n A}(p_n - \gamma_n L^* v_n^*), \quad b_n := J_{\mu_n B}(Lp_n + \mu_n v_n^*), \\ a_n^* &:= \gamma_n^{-1}(p_n - a_n) - L^* v_n^*, \quad b_n^* := \mu_n^{-1}(Lp_n - b_n) + v_n^*, \end{aligned}$$

It easy to see $H_{\varphi_n} = H_{a_n,b_n^*}$, where $\varphi_n = \varphi(a_n, b_n^*)$.

Best approximation iterative schemes

For any $x, y \in \mathcal{H} \times \mathcal{G}$ we define

$$H(x, y) := \{h \in \mathcal{H} \times \mathcal{G} \mid \langle h - y \mid x - y \rangle \leq 0\}.$$

As previously, let $\{H_n\}_{n \in \mathbb{N}} \subset \mathcal{H} \times \mathcal{G}$ be a sequence of closed convex sets, $Z \subset H_n$ for $n \in \mathbb{N}$.

Algorithm 2 Generic primal-dual best approximation iterative scheme

Choose an initial point $x_0 = (p_0, v_0^*) \in \mathbf{H} \times \mathbf{G}$
Choose a sequence of parameters $\{\lambda_n\}_{n \geq 0} \in (0, 1]$
for $n = 0, 1 \ldots$ **do**
  Fejérian step
  $x_{n+1/2} = x_n + \lambda_n (P_{H_n}(x_n) - x_n)$
  Let C_n be a closed convex set such that $Z \subset C_n \subset H(x_n, x_{n+1/2})$.
  Haugazeau step
  $x_{n+1} = P_{H(x_0, x_n) \cap C_n}(x_0)$
end for

The choice of $C_n = H(x_n, x_{n+1/2})$ has been already investigated in [1].

Convergence

Theorem

Let Z be a nonempty closed convex subset of $H \times G$ and let $x_0 = (p_0, v_0^*) \in \mathbf{H} \times \mathbf{G}$. Let $\{C_n\}_{n\in\mathbb{N}}$ be any sequence satisfying $Z \subset C_n \subset H(x_n, x_{n+1/2})$, $n \in \mathbb{N}$. For the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by Iterative Scheme 2 the following hold:

1. $Z \subset H(x_0, x_n) \cap C_n$ *for* $n \in \mathbb{N}$,
2. $\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$ *for* $n \in \mathbb{N}$,
3. $\sum_{n=0}^{+\infty} \|x_{n+1} - x_n\|^2 < +\infty$,
4. $\sum_{n=0}^{+\infty} \|x_{n+1/2} - x_n\|^2 < +\infty$.
5. *If*
$$\forall x \in H \times G \ \forall \{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N} \quad x_{k_n} \rightharpoonup x \implies x \in Z,$$
then $x_n \to P_Z(x_0)$.

Best approximation algorithm

Yves. Haugazeau. "Sur les inequations variationnelles et la minimisation de fonctionnelles convexes". French. PhD thesis. [S.l.]: [s.n.], 1968

Heinz H. Bauschke and Patrick L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. With a foreword by Hédy Attouch. Springer, New York, 2011, pp. xvi+468. ISBN: 978-1-4419-9466-0. DOI: 10.1007/978-1-4419-9467-7. URL: http://dx.doi.org/10.1007/978-1-4419-9467-7

Abdullah Alotaibi, Patrick L. Combettes, and Naseer Shahzad. "Best approximation from the Kuhn-Tucker set of composite monotone inclusions". In: *Numer. Funct. Anal. Optim.* 36.12 (2015), pp. 1513–1532. ISSN: 0163-0563. DOI: 10.1080/01630563.2015.1077864. URL: http://dx.doi.org/10.1080/01630563.2015.1077864

Generic best approximation algorithm

Let $x_0 = (u_0, v_0^*) \in \mathcal{H} \times \mathcal{G}$.

1: **for** $n = 0, 1, \ldots$ **do**
2: $\quad (u_{n+1}, v_{n+1}^*) = P_{H_1(x_0,(u_n,v_n^*)) \cap H_2(u_n,v_n^*)}(x_0)$
3: **end for**

where

$$H_1(x_0, (u, v^*)) := \{h \in \mathcal{H} \times \mathcal{G} \mid \langle h - (u, v^*) \mid x_0 - (u, v^*)\rangle \leq 0\},$$
$$H_2(u, v^*) := \{h \in \mathcal{H} \times \mathcal{G} \mid \langle h \mid g(u, v^*)\rangle \leq f(u, v^*)\}, \quad Z \subset H_2(u, v^*),$$
$$g : \mathcal{H} \times \mathcal{G} \to \mathcal{H} \times \mathcal{G}, \quad f : \mathcal{H} \times \mathcal{G} \to \mathbb{R}$$

for suitably chosen f and g.

$$Z \subset H_1(x_0, (u_n, v_n^*)) \cap H_2(u_n, v_n^*) \implies Z \subset H_1(x_0, (u_{n+1}, v_{n+1}^*)), \quad n \in \mathbb{N}$$

Special case with resolvents

Let $x_0 = (u_0, v_0^*) \in \mathcal{H} \times \mathcal{G}$.

for $n = 0, 1, \ldots$ **do**
  Pick $(\gamma_n, \mu_n) \in [\varepsilon, 1/\varepsilon]^2$
  $a(u_n, v_n^*) = J_{\gamma_n A}(u_n - \gamma_n L^* v_n^*),\ a^*(u_n, v_n^*) = \frac{1}{\gamma_n}(u_n - \gamma_n L^* v_n^* - a(u_n, v_n^*))$
  $b(u_n, v_n^*) = J_{\mu_n B}(L u_n + \mu_n v_n^*),\ b^*(u_n, v_n^*) = \frac{1}{\mu_n}(L u_n + \mu_n v_n^* - b(u_n, v_n^*))$
  if $s(u_n, v_n^*) = 0$ **then**
    $\bar{u} = a_n,\ \bar{v}^* = b_n^*, \quad (\bar{u}, \bar{v}^*) \in Z$
    terminate
  else
    $(u_{n+1}, v_{n+1}^*) = P_{H_1(x_0,(u_n,v_n^*)) \cap H_2(u_n,v_n^*)}(u_0, v_0^*)$
  end if
end for

$$\text{where} \quad s(u_n, v_n^*) := \begin{bmatrix} a^*(u_n, v_n^*) + L^* b^*(u_n, v_n^*) \\ b(u_n, v_n^*) - L a(u_n, v_n^*) \end{bmatrix},$$

$$\eta(u_n, v_n^*) := \langle a(u_n, v_n^*) \mid a^*(u_n, v_n^*) \rangle + \langle b(u_n, v_n^*) \mid b^*(u_n, v_n^*) \rangle,$$

$$H_2(u_n, v_n^*) := \{ h \in \mathcal{H} \times \mathcal{G} \mid \langle h \mid s(u_n, v_n^*) \rangle \leq \eta(u_n, v_n^*) \}.$$

Theorem (Alatoibi, Combettes, Shahzad, 2015)

$(u_n)_{n\in\mathbb{N}}$ converges strongly to a point $\bar{u}$, $(v_n)_{n\in\mathbb{N}}$ converges strongly to a point $\bar{v}^*$ and $(\bar{u}, \bar{v}^*) = P_Z(x_0)$.

Lipschitzness of data

1. Let $(u, v^*) \in D := \mathrm{cl}B(\frac{x_0+P_Z(x_0)}{2}, \frac{\|x_0-P_Z(x_0)\|}{2})$. Then $P_Z(x_0) \in H_1(x_0, (u, v^*))$ and for all $(u, v^*) \notin D$, $P_Z(x_0) \notin H_1(x_0, (u, v^*))$
2. Let $\gamma, \mu \in \mathbb{R}_{++}$. Operator $\eta : D \to \mathbb{R}$, defined as

$$\eta(u, v^*) := \langle J_{\gamma A}(p - \gamma L^* v) \mid \frac{1}{\gamma}(u - \gamma L^* v - J_{\gamma A}(u - \gamma L^* v))\rangle + \langle J_{\mu B}(Lu + \mu v^*) \mid L^*(Lu + \mu v^*)\rangle$$

   is Lipschitz continuous on D.
3. Let $\gamma, \mu \in \mathbb{R}_{++}$. An operator $s : \mathcal{H} \times \mathcal{G} \to \mathcal{H} \times \mathcal{G}$ defined as

$$s(u, v^*) := \begin{bmatrix} \frac{1}{\gamma}(u - \gamma L^* v - J_{\gamma A}(u - \gamma L^* v)) + \frac{1}{\mu} L^*(Lu + \mu v^* - J_{\mu B}(Lu + \mu v^*)) \\ J_{\mu B}(Lu + \mu v^*) - L J_{\gamma A}(u - \gamma L^* v) \end{bmatrix}$$

   is Lipschitz continuous on $\mathcal{H} \times \mathcal{G}$.
4. $$H_1(x_0, (u, v^*)) \cap H_2(u, v^*) = \left\{ x \in \mathcal{H} \times \mathcal{G} \;\middle|\; \begin{array}{rcl} \langle x \mid x_0 - (u, v^*)\rangle & \le & \langle (u, v^*) \mid x_0 - (u, v^*)\rangle, \\ \langle x \mid s(u, v^*)\rangle & \le & \eta(u, v^*) \end{array} \right\}$$

Optimization and Dynamical Systems

Dynamical systems and iterative solution schemes

1. U. Helmke and J. Moore. Optimization and dynamical systems (1996),A. Taylor, B. Van Scoy, L. Lessard Lyapunov Functions for First-Order Methods: Tight Automated Convergence Guarantees (2015), J. Schropp and I. Singer. A dynamical systems approach to constrained minimization, (2000)
2. R.I. Bot, E.R. Csetnek, A dynamical system associated with the fixed point set ofa nonexpansive operator (2018), A.S.Antipin, Minimization of convex functions on convex sets by means of differential equations (1996)
3. B.Abbas, H. Attouch, Dynamical systems and forward-backward algorithms associated with the sumof a convex subdifferential and a monotone cocoercive operator (2014), B. Abbas h. Attouch B.F. Svaiter, Newton-like dynamics and forward-backward methods for structured monotone inclusions in hilbert spaces (2014), H. Attouch, M.-O. Czarnecki, Asymptotic behavior of coupled dynamical systems with multiscale aspects (2011)
4. S. Boyd, W. Su, E.J. Candés, A Differential Equation for Modeling Nesterov's Accelerated Gradient Method; Theory and Insights (2014)

Modeling Nesterov's accelerated gradient method

1. starting x_0, $y_0 = x_0$

$$x_k := y_{k_1} - s\nabla f(y_{k-1}), \quad y_k := x_k + \frac{k-1}{k+2}(x_k - x_{k-1}).$$

2. combining the two with a rescaling we get

$$\frac{x_{k+1} - x_k}{\sqrt{s}} = \frac{x_{k+1} - x_k}{\sqrt{s}} - \sqrt{s} f(y_{k-1}) \tag{2}$$

3. $x_k \approx X(k\sqrt{s})$ for some smooth curve $X(t)$, $t \geq 0$, we put $k = t/\sqrt{s}$
4. as $s \to 0$, then $X(t) \approx x_{t/\sqrt{s}} = x_k$ and $X(t+\sqrt{s}) \approx x_{(t+\sqrt{s})/\sqrt{s}} = x_{k+1}$
5. by the Taylor expansion

$$\frac{x_{k+1} - x_k}{\sqrt{s}} = \dot{X}(t) + \frac{1}{2}\ddot{X}(t)\sqrt{s} + o(\sqrt{s}), \quad \frac{x_k - x_{k-1}}{\sqrt{s}} = \dot{X}(t) - \frac{1}{2}\ddot{X}(t)\sqrt{s} + o(\sqrt{s})$$

6. together with $\sqrt{s}\nabla f(y_k) = \sqrt{s}\nabla f(X(t)) + o(\sqrt{s})$ the formula (2) gives

$$\ddot{X} + \frac{3}{t}\dot{X} + \nabla f(X) = 0$$

with $X(0) = x_0$, $\dot{X}(0) = 0$.

Projected Dynamical System

J.P. Aubin and A. Cellina. *Differential Inclusions: Set-Valued Maps and Viability Theory*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 1984. ISBN: 9783540131052. URL: https://books.google.it/books?id=KDqXQgAACAAJ

Paul Dupuis and Anna Nagurney. "Dynamical systems and variational inequalities". In: *Annals of Operations Research* 44.1 (1993), pp. 7–42. ISSN: 1572-9338. DOI: 10.1007/BF02073589. URL: https://doi.org/10.1007/BF02073589

Anna Nagurney and Ding Zhang. *Projected Dynamical Systems and Variational Inequality with Applications*. Vol. 2. Jan. 1996

M G. Cojocaru and L B. Jonker. "Existence of solutions to projected differential equations in Hilbert spaces". In: 132 (Jan. 2004)

Hedy Attouch and Felipe Alvarez. "The Heavy Ball With Friction Dynamical System for Convex Constrained Minimization Problems". In: (Feb. 2001)

Radu Ioan Boț and Ernö Robert Csetnek. "Convergence rates for forward-backward dynamical systems associated with strongly monotone inclusions". In: *J. Math. Anal. Appl.* 457.2 (2018), pp. 1135–1152. ISSN: 0022-247X. DOI: 10.1016/j.jmaa.2016.07.007. URL: https://doi.org/10.1016/j.jmaa.2016.07.007

Weijie Su, Stephen Boyd, and Emmanuel J. Candès. "A differential equation for modeling Nesterov's accelerated gradient method: theory and insights". In: *J. Mach. Learn. Res.* 17 (2016), Paper No. 153, 43. ISSN: 1532-4435

Projected Dynamical System and Differential Variational Inclusion

The *differential projection* of the vector h at $x \in D \subset \mathcal{H} \times \mathcal{G}$ with respect to the set D

$$\Pi_K(x, h) = \lim_{\Delta t \to 0} \frac{P_D(x + \Delta t h) - x}{\Delta t}.$$

A *projected dynamical system* (PDS) takes the form

$$\begin{aligned} &\dot{x}(t) = \Pi_D(x(t), F(x(t))) = F(x(t)), \quad \text{for a.a } t \geq 0, \\ &x(0) = x_0 \in D, \end{aligned} \tag{PDS}$$

where D — bounded closed convex set, $F: \ D \to \mathcal{H} \times \mathcal{G}$ — vector field defined as

$$F(x(t)) := P_{H_1(x_0, x(t)) \cap H_2(x(t))}(x_0) - x(t).$$

(PDS) is a particular case of a differential variational inclusion (DVI) given as follows

$$\begin{aligned} &\dot{x}(t) \in F(x(t)) - N_D(x(t)), \quad \text{for a.a } t \geq 0, \\ &x(0) = x_0. \end{aligned} \tag{DVI}$$

We are interested in finding an absolutely continuous function $x: \ \mathbb{R}_+ \to D$ satisfying (PDS).[1]

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[1](PDS) consists in finding the *slow* solution (solution of the minimal norm) of (DVI).

J. Gwinner. "On differential variational inequalities and projected dynamical systems—equivalence and a stability result". In: *Discrete Contin. Dyn. Syst.* Dynamical systems and differential equations. Proceedings of the 6th AIMS International Conference, suppl. (2007), pp. 467–476. ISSN: 1078-0947

Discretization of (PDS) - relation to Best Approximation Algorithm

Discretization of (PDS) with respect to time variable t and step size $1 \geq h_n > 0$

$$\frac{x_{n+1} - x_n}{h_n} = P_{H_1(x_0, x_n) \cap H_2(x_n)}(x_0) - x_n. \tag{3}$$

Taking stepsizes $h_n = 1$ gives

$$x_{n+1} = P_{H_1(x_0, x_n) \cap H_2(x_n)}(x_0). \tag{4}$$

This shows that the best approximation sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfies the discretized (PDS) .

Existence results for PDS

Theorem (Cojocaru, Jonker)

Let $F : D \to \mathcal{H} \times \mathcal{G}$ be a Lipschitz continuous vector field with Lipschitz constant b. Let $x_0 \in D$ and $r > 0$ such that $\|x\| \leq r$. Then the initial value problem $\frac{dx(t)}{dt} = \Pi_D(x(t); F(x(t)))$, $x(0) = x_0 \in D$ has a unique solution on the interval $[0, \ell]$, where $\ell := \frac{r}{\|F(x_0)\| + bL}$.

Monica-Gabriela Cojocaru and Leo B. Jonker. "Existence of solutions to projected differential equations in Hilbert spaces". In: *Proc. Amer. Math. Soc.* 132.1 (2004), pp. 183–193. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-03-07015-1. URL: https://doi.org/10.1090/S0002-9939-03-07015-1

Theorem (Cojocaru)

Under assumptions of the above Theorem for the initial value problem $\frac{dx(t)}{dt} = \Pi_K(x(t); F(x(t)))$, $x(0) = x_0 \in D$ the solutions can be extended to $[0, +\infty)$

Monica Gabriela Cojocaru. *Projected dynamical systems on Hilbert spaces*. Thesis (Ph.D.)–Queen's University (Canada). ProQuest LLC, Ann Arbor, MI, 2002, p. 89. ISBN: 978-0612-73289-6

Problem: Lipschitzness of projection $G(x(t)) = P_{H_1(x_0, x(t)) \cap H_2(x(t))}(x_0)$.

Lipschitzness of projection onto moving polyhedral sets

Let $\mathcal{X}$ be a Hilbert space and $D \subset \mathcal{X}$. Let $H: D \rightrightarrows \mathcal{X}$ set-valued mapping given as

$$H(p) := \left\{ x \in \mathcal{X} \mid \begin{array}{ccc} \langle x \mid g_i(p) \rangle & = & f_i(p), \quad i \in I_1 \\ \langle x \mid g_i(p) \rangle & \leq & f_i(p), \quad i \in I_2 \end{array} \right\}, \ H(p) \neq \emptyset, \ p \in D,$$

where $f_i(p): D \to \mathbb{R}$, $g_i(p): D \to \mathcal{X}$, $i \in I_1 \cup I_2$ are Lipschitz functions.

Let $x_0 \in D$. For $p \in D$ the function $G(p) = P_{H(p)}(x_0)$ is well defined.

Finding $P_{H(p)}(x_0)$ is equivalent to find $y \in H(p)$ solving variational inequality

$$\langle x_0 - y \mid x - y \rangle \leq 0 \quad \text{for all } x \in H(p). \tag{VI}$$

Let

$$P(x_0, p) := \{ x \in \mathcal{X} \mid x_0 \in x + \partial_x h(x, p) \},$$

where stands $\partial_x h(x, p)$ for the partial limiting subdifferential of h with respect to x. If $h(x, p) = \iota_{H(p)}(x)$, where ι is the indicator function of $H(p)$, then

$$P(x_0, p) = P_{H(p)}(x_0) = (N_{H(p)} + I)^{-1}(x_0),$$

where $N_{H(p)}$ is the normal cone to $H(p)$. The case where

$$H(p) := \{ x \in \mathbb{R}^n \mid \ \langle x \mid g_i \rangle \ \leq \ f_i(p), \quad i \in I_2 \ \}, \ H(p) \neq \emptyset, \ p \in D,$$

was investigated e.g. in

N. D. Yen. "Lipschitz Continuity of Solutions of Variational Inequalities with a Parametric Polyhedral Constraint". In: *Mathematics of Operations Research* 20.3 (1995), pp. 695–708. ISSN: 0364765X, 15265471. URL: http://www.jstor.org/stable/3690178

General parametric variational system

General parametric variational system of finding $x \in \mathcal{X}$

$$v \in f(x,p,q) + \partial_x h(x,p), \quad \text{for } p \in \mathcal{P},\ q \in \mathcal{Q}, v \in \mathcal{X},$$

p, q - parameters, $\partial_x h$ stands for the partial limiting subdifferential of the function h with respect to variable x. For $h(x,p) = \iota_{H(p)}(x)$ and $f(x,p,q) = x$, $v = x_0$.

Theorem (Mordukhovich, Nghia, Pham)

Let $\bar{p} \in D$ and hypotheses (A2) and (A3) of paper [1] be satisfied. The following are equivalent:

(i) *There exists a neighbourhood V of x_0 a neighbourhood U of $\bar{p}$ such that*

$$\|(v_1 - v_2) - 2\kappa_0(P_{H(p_1)}(v_1) - P_{H(p_2)}(v_2))\| \leq \|v_1 - v_2\| + \ell_0\|p_1 - p_2\|$$

holds for all $(v_1, p_1), (v_2, p_2) \in V \times U$ with some positive constants κ_0 and ℓ_0.

(ii) *Graphical subdifferential mapping*

$$R: \ p \to gphN_{H(p)}(\cdot)$$

is Lipschitz-like around $(\bar{p}, H(\bar{p}), x_0 - H(\bar{p}))$, where $N_{H(p)}(x)$ is the normal cone to $H(p)$ at $x \in H(p)$.

[1] B. S. Mordukhovich, T. T. A. Nghia, and D. T. Pham. "Full Stability of General Parametric Variational Systems". In: *Set-Valued and Variational Analysis* (2018). ISSN: 1877-0541. DOI: 10.1007/s11228-018-0474-7

Relaxed constant rank constraint qualification (RCRCQ)

For any $(p, x) \in D \times \mathcal{H}$ let $I_p(x) := \{i \in I_1 \cup I_2 \mid \langle x \mid g_i(p) \rangle = f_i(p)\}$ be the active index set for $p \in D$ at $x \in \mathcal{H}$.

Definition

The *Relaxed constant rank constraint qualification* (RCRCQ) is satisfied in $(\bar{x}, \bar{p})$, $\bar{x} \in H(\bar{p})$, if there exists a neighbourhood $U(\bar{p})$ of $\bar{p}$ such that, for any index set J, $I_1 \subset J \subset I_{\bar{p}}(\bar{x})$, for every $p \in U(\bar{p})$ the system of vectors $\{g_i(p), i \in J\}$ has constant rank. Precisely, for any J, $I_1 \subset J \subset I_{\bar{p}}(\bar{x})$

$$\operatorname{rank}(g_i(p,), i \in J) = \operatorname{rank}(g_i(\bar{p}), i \in J) \quad \text{for all } p \in U(\bar{p}).$$

L. Minchenko and S. Stakhovski. "Parametric Nonlinear Programming Problems under the Relaxed Constant Rank Condition". In: *SIAM Journal on Optimization* 21.1 (2011), pp. 314–332. DOI: 10.1137/090761318. eprint: https://doi.org/10.1137/090761318. URL: https://doi.org/10.1137/090761318

This definition has been introduced in finite dimensional case by Minchenko and Stakhovski for more general set-valued mappings

$$H(p) := \left\{ x \in \mathcal{X} \mid \begin{array}{ccc} \xi_i(p,x) & = & 0, & i \in I_1 \\ \xi_i(p,x) & \leq & 0, & i \in I_2 \end{array} \right\},$$

where $\xi_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in I_1 \cup I_2$ are continuously differentiable functions with respect to variable x. Non-parametric versions of constant rank qualifications has been studied by Kruger and Minchenko and Outrata, Andreani and Haeser and Schuverdt and Silva.

R-regularity of a set-valued mapping

Let $\mathbb{C}: \ D \rightrightarrows \mathcal{H}$ be a multifunction defined as $\mathbb{C}(p) := C(p)$, where

$$C(p) = \left\{ x \in \mathcal{H} \;\middle|\; \begin{array}{ll} \langle x \mid g_i(p) \rangle = f_i(p), & i \in I_1, \\ \langle x \mid g_i(p) \rangle \leq f_i(p), & i \in I_2 \end{array} \right\}, \qquad (5)$$

and $f_i: \ D \to \mathbb{R}$, $g_i: \ D \to \mathcal{H}$, $i \in I_1 \cup I_2$, $I_1 = \{1, \ldots, m\}$, $I_2 = \{m+1, \ldots, n\}$ are Lipschitz on D with Lipschitz constants ℓ_{f_i}, ℓ_{g_i}, respectively.

Definition

Multifunction $\mathbb{C}: \ D \rightrightarrows \mathcal{H}$ given by (5) is said to be R-regular at a point $(\bar{p}, \bar{x})$, if for all (p, x) in a neighbourhood of $(\bar{p}, \bar{x})$,

$$dist\,(x, \mathbb{C}(p)) \leq \alpha \max\{0, \ |\langle x \mid g_i(p)\rangle - f_i(p)|, \ i \in I_1, \ \langle x \mid g_i(p)\rangle - f_i(p), \ i \in I_2\}$$

for some $\alpha > 0$.

Theorem

Let $\mathcal{H}$, $\mathcal{G}$ be a Hilbert spaces and $f_i: \ D \to \mathbb{R}$, $g_i: \ tD \to \mathcal{H}$ are Lipschitz on $D \subset \mathcal{G}$. If the set-valued mapping $\mathbb{C}: \ D \rightrightarrows \mathcal{H}$ given by (5) is R-regular at $(\bar{p}, \bar{x})$, $\bar{p} \in D$, $\bar{x} \in C(\bar{p})$ then $\mathbb{C}$ is Lipschitz-like at $(\bar{p}, \bar{x})$

Lipschitz likeness of the graphical subdifferential mapping

Let $\bar{p} \in D$. The lower Kuratowski limit is defined as

$$\liminf_{p \to \bar{p}} H(\bar{p}) := \{x \in \mathcal{X} \mid \forall p_k \to \bar{p} \; \exists \; x_k \in H(p_k) \quad s.t. \; x_k \to x\}$$

and $G(\bar{p}) = P_{H(\bar{p})}(x_0)$.

Theorem (Main result 1)

Let $\mathcal{X}$ be a Hilbert space, $\bar{p} \in D \subset \mathcal{H}$ and

$$H(p) := \left\{ x \in \mathcal{X} \mid \begin{array}{lcll} \langle x \mid g_i(p) \rangle & = & f_i(p), & i \in I_1 \\ \langle x \mid g_i(p) \rangle & \leq & f_i(p), & i \in I_2 \end{array} \right\},$$

where $f_i(p) : \; D \to \mathbb{R}$, $g_i(p) : \; D \to \mathcal{X}$ are Lipschitz functions. Suppose that $x_0 \notin H(\bar{p})$, $G(\bar{p}) \in \liminf_{p \to \bar{p}} H(p)$ and RCRCQ holds at $(G(\bar{p}), \bar{p})$. Then the graphical subdifferential mapping

$$R : \; p \to gphN_{H(p)}(\cdot)$$

is Lipschitz-like around $(\bar{p}, G(\bar{p}), x_0 - G(\bar{p}))$

Remark

Under the assumption of RCRCQ hypotheses (A2) and (A3) of Theorem Mordukhovich, Nghia, Pham hold.

Lipschitzness of the projection

Consequence of main theorem 1 and theorem of Mordukhovich, Nghia, Pham

Theorem (Main result 2)

Let $\mathcal{X}$ be a Hilbert space and $\bar{p} \in D \subset \mathcal{X}$. Assume that RCRCQ holds at $(G(\bar{p}), \bar{p})$ and $G(\bar{x}) \in \liminf_{p \to \bar{p}} H(p)$. Then there exists a neighbourhood U of $\bar{p}$ and a constant $\ell_0 > 0$ such that

$$\|G(p_1) - G(p_2)\| \leq \ell_0 \|p_1 - p_2\|, \quad (p_1, p_2) \in U.$$

Consequence for the vector field related to proximal primal-dual dynamical system

Consequence for the vector field related to proximal primal-dual dynamical system

Let $I_1 = \emptyset$, $I_2 = \{1,2\}$, $\bar{z} = P_Z(x_0)$. Let $x_0 \in \mathcal{H} \times \mathcal{G}$, $\bar{p} \in D \setminus \{x_0, P_Z(x_0)\}$ and

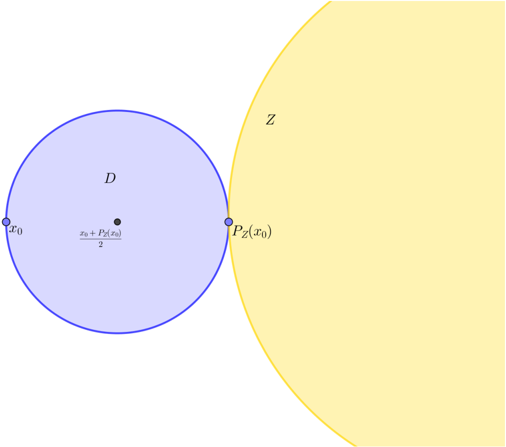
$$G(p) := G(u,v^*) := H_1(x_0,(u,v^*)) \cap H_2(u,v^*) =$$

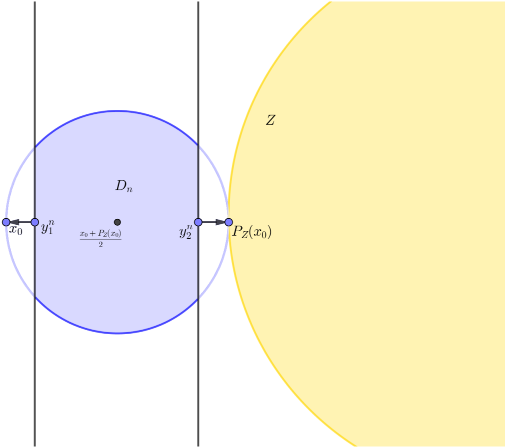
$$\left\{ x \in \mathcal{H} \times \mathcal{G} \;\middle|\; \begin{array}{rcl} \langle x \mid x_0 - (u,v^*)\rangle & \leq & \langle (u,v^*) \mid x_0 - (u,v^*)\rangle, \\ \langle x \mid s(u,v^*)\rangle & \leq & \eta(u,v^*) \end{array} \right\}$$

Let $g_1(p) := g_1(u,v^*) := x_0 - (u,v^*)$, $g_2(p) := g_2(u,v^*) := s(u,v^*)$. Then $g_i(\bar{p})$, $i \in I_{\bar{p}}(H(\bar{p}))$ are linearly independent and moreover set-valued mapping H satisfies RCRCQ at $(H(\bar{p}), \bar{p})$.

Remark

Let us note that $g_1(x_0) = g_1(u_0, v_0^*) = 0$ *and* $g_2(P_Z(x_0)) = g_2(\bar{u}, \bar{v}^*) = s(\bar{u}, \bar{v}^*) = 0$ *and at points* $x_0, P_Z(x_0)$ *RCRCQ does not hold.*


Z
D
x_0
$\frac{x_0 + P_Z(x_0)}{2}$
$P_Z(x_0)$



$$y_1^n := x_0 + \varepsilon_n \frac{P_Z(x_0) - x_0}{\|P_Z(x_0) - x_0\|},$$

$$y_2^n := P_Z(x_0) + \varepsilon_n \frac{x_0 - P_Z(x_0)}{\|x_0 P_Z(x_0)\|},$$

$$\varepsilon_n > 0, \quad D_n := D \cap H_1(y_1^n, x_0) \cap H_1(y_2^n, P_Z(x_0)).$$

Consequences

Let $\mathcal{H} \times \mathcal{G}$ be finite-diminensional Hilbert space. Let $\varepsilon_n = \frac{1}{n}$ and $D_n \subset D \subset \mathcal{H} \times \mathcal{G}$ be defined as before. Then $D = \overline{\bigcup_n D_n}$ and for any $n \in \mathbb{N}$, F is Lipschitz continuous on D_n. Hence projected dynamical system

$$\begin{aligned} &\dot{x}(t) = \Pi_{D_n}(x(t), F(x(t))), \quad \text{for a.a } t \geq 0, \\ &x(0) = x_{0,n}, \quad x(t) \in D_n \quad \text{for a.a } t \geq 0 \end{aligned} \qquad (BA - PDS_n)$$

is uniquely solvable for each n and the solution $x(t)$ of $(BA - PDS_n)$ is absolutely continuous.

J. Gwinner. "On differential variational inequalities and projected dynamical systems—equivalence and a stability result". In: *Discrete Contin. Dyn. Syst.* Dynamical systems and differential equations. Proceedings of the 6th AIMS International Conference, suppl. (2007), pp. 467–476. ISSN: 1078-0947

Thank you for your attention!