Metric Regularity and Directional Metric Regularity of Multifunctions and Applications

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Outline

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Metric Regularity

Consider an equation of the form

$$F(x) = y, \qquad (1)$$

where $F : X \to Y$ is a function, X, Y are metric spaces.

The distance $d(y, F(x))$ is used to judge approximate solutions. The error of some approximate solution x is

$$d(x, F^{-1}(y)) = \inf\{d(x, u) : \ F(u) = y\}.$$

One seeks so an error bound of the form

$$d(x, F^{-1}(y)) \leq Kd(y, F(x)) \qquad (2)$$

for all (x, y) globally, or locally, that is, (x, y) near a given $(\bar{x}, \bar{y})$ with $\bar{y} = F(\bar{x})$, and F is said to metrically regular at $\bar{x}$. The infimum of such K is regular modulus: $\mathrm{reg}\, F(\bar{x})$.

Generalized Equations

When Y is a m-dimensional space, we often deal with a system of inequalities:

$$F_i(x) \leq y_i, i = 1, ..., m. \quad (3)$$

Such inequalities systems are used in optimization for problems with inequalities constraints. This system of inequalities can be studied via the generalized equation : $y \in F(x)$, where,

$$F(x) := (F_i(x))_{i=1,\ldots,m} + \mathbb{R}^m_+; \quad y = (y_i)_{i=1,\ldots,m}, \quad (4)$$

then $F : X \rightrightarrows \mathbb{R}^m$ is a multifunction.

A multifunction (Set-valued) is regular at $(\bar{x}, \bar{y})$ ($\bar{y} \in F(\bar{x})$) if

$$d(x, F^{-1}(y)) \leq Kd(y, F(x)) \quad \text{for all} \quad (x, y) \quad \text{near} \quad (\bar{x}, \bar{y}).$$

Banach-Schauder open mapping theorem

X, Y : Banach spaces; $A \in \mathcal{L}(X, Y)$

If $\mathrm{Im}\ A = Y$ *then* A *is open:* $\exists r > 0$ *such that*

$$rB_Y \subseteq A(B_X).$$

The upper bound of such r *is the Banach constant of* A :

$$C(A) = \inf\{\|A^* y^*\| : \ \|y^*\| = 1\}.$$

Moreover,

$$d(x, A^{-1}(y)) \leq C(A)^{-1}\|Ax - y\| \quad \text{for all } (x, y) \in X \times Y.$$

Lusternik-Graves theorem

X, Y : Banach spaces; $F : X \to Y$ continuously differentiable at $\bar{x}$; $F(\bar{x}) := \bar{y}$.

If $\mathrm{Im}\ F'(\bar{x}) = Y$ *then* $\exists r > 0,\ \exists \varepsilon > 0$:

$$B(\bar{y}, rt) \subseteq F(B(\bar{x}, t)) \ \ \forall t \in (0, \varepsilon).$$

The upper bound of such r *is the Banach constant of* $F'(\bar{x})$ *is* $C(F'(\bar{x}))$, *the Banach constant of* $F'(\bar{x})$. *Moreover,*

$$d(x, F^{-1}(y)) \leq r^{-1} d(y, F(x)) \quad \text{for all } (x, y) \text{ near } (\bar{x}, \bar{y}).$$

Robinson and Mangasarian-Fromovitz constraint qualifications

$F := g - C$, $g : X \to Y$ is of $\mathcal{C}^1$ class; $C \subseteq Y$ is a nonempty closed convex subset. Given $(\bar{x}, 0) \in \mathrm{gph}\ F$,

- F is metrically regular at $(\bar{x}, 0)$ $\iff$ Robinson constraint qualification (RCQ):

$$0 \in \mathrm{int}[g(\bar{x}) + Dg(\bar{x})X - C].$$

- System of equality and inequality: $(\mathrm{RCQ}) \quad \Leftrightarrow \quad (MFCQ)$ (Mangasarian-Fromovitz constraint qualification)

Robinson-Ursescu Theorem

When F has a closed and convex graph, the *Robinson-Ursescu Theorem* says that F is metrically regular at (x_0, y_0) if and only if $y_0 \in \mathrm{int}(\mathrm{Im} F)$.

Setting of metric spaces

X: (complete) metric space

Y : metric space

$F : X \rightrightarrows Y$ multifunction (set-valued mapping) (which associates with every $x \in X$ a set $F(x) \subseteq Y$)

$\mathrm{gph}\ F := \{(x, y) \in X \times Y : \ y \in F(x)\}$

$F^{-1} : Y \rightrightarrows X,\ F^{-1}(y) = \{x \in X : \ \ y \in F(x)\}$

$(\bar{x}, \bar{y}) \in \mathrm{gph}\ F$ is given

Definitions of regularity

- **metric regularity**: $\exists K > 0, \varepsilon > 0$ s.t.

$$d(x, F^{-1}(y)) \le K d(y, F(x)), \quad \forall (x,y) \in B((\bar{x},\bar{y}),\varepsilon),$$

$\mathrm{reg}\, F(\bar{x},\bar{y})$:= infimum of such K : *the rate of metric regularity.*

- **openness at a linear rate**: $\exists r, \varepsilon > 0$ s.t.

$$B(y, tr) \subseteq F(B(x,t)), \quad \forall (x,y) \in B((\bar{x},\bar{y}),\varepsilon) \cap \mathrm{gph}\ F,$$

$\mathrm{sur}\, F(\bar{x},\bar{y})$:= supremum of such r : *rate of openness (or surjection).*

- **Lipschitz-like (or Aubin) property**: $\exists K, \varepsilon > 0$ s.t.

$$d(y, F(x)) \le K d(x,u), \quad \forall x \in B(\bar{x},\varepsilon),\ (u,y) \in B((\bar{x},\bar{y}),\varepsilon) \cap \mathrm{gph}\ F,$$

$\mathrm{lip}\, F(\bar{x},\bar{y})$:= supremum of such K : *Lipschitz rate.*

Equivalence.

Under the convention $1/\infty = 0$, *one has*

$$\operatorname{reg} F(\bar{x}, \bar{y}) = \operatorname{lip} F^{-1}(\bar{y}, \bar{x}) = \frac{1}{\operatorname{sur} F(\bar{x}, \bar{y})}.$$

(cf. Borwein-Zuang 1988, Kruger 1988 Penot 1989, also Ioffe 1981)

- F is said to be *regular* at $(\bar{x}, \bar{y})$ if (one of) the three properties hold.

Characterization of regularity

- Set
$$\varphi_y(x) = \varphi(x, y) = \liminf_{u \to x} d(y, F(u)),$$
the *lower semicontinuous envelope* of the distance function $d(y, F(\cdot))$.

- **General characterization of regularity.** (Ngai-Théra- 2008) *Suppose that* $\operatorname{gph} F$ *is closed. Then F is regular at* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *with* $\operatorname{sur} F(\bar{x}, \bar{y}) > r > 0$ *iff for any* (x, y) *in a neighborhood of* $(\bar{x}, \bar{y})$ *with* $y \notin F(x)$, *we can find* $u \in X$ *s.t.*
$$rd(u, x) < \varphi(x, y) - \varphi(u, y).$$

Infinitesimal characterization: slopes

- **Definition** (Degiorgi-Marino-Tosques (1980)) Slope of a lower semicontinuous function f at $x \in \mathrm{dom} f$ is the quantity defined by $|\nabla f|(x) = 0$ if x is a local minimum of f, otherwise

$$|\nabla f|(x) = \limsup_{y \to x, y \neq x} \frac{f(x) - f(y)}{d(x,y)}.$$

For $x \notin \mathrm{dom} f$, we set $|\nabla f|(x) = +\infty$.

- **Example.** X, Y are normed spaces, $f \in C^1 : |\nabla f|(x) = \|f'(x)\|$.

Infinitesimal regularity criteration

Theorem. (N.-Tron-Théra - 2011) *X: complete metric space; Y : metric space; $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \mathrm{gph}\, F$. Then*

$$\mathrm{sur}\, F(\bar{x}, \bar{y}) \geq \liminf_{(x,y)\to(\bar{x},\bar{y}),\, y \notin F(x)} |\nabla \varphi_y|(x).$$

Moreover, if Y is a normed space (or more general, a smooth manifold, or a length metric space) then

$$\mathrm{sur}\, F(\bar{x}, \bar{y}) = \liminf_{(x,y)\to(\bar{x},\bar{y}),\, y \notin F(x)} |\nabla \varphi_y|(x).$$

An application: Lipschitz perturbation

Milyutin perturbation theorem. *Y: normed space;* $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$; $g : X \to Y$ *is Lipschitz near* $\bar{x}$. *Then*

$$\operatorname{sur}(F + g)(\bar{x}, \bar{y} + g(\bar{x})) \geq \operatorname{sur} F(\bar{x}, \bar{y}) - \operatorname{lip} g(\bar{x}).$$

Fréchet Subdifferential - Normal cone - Coderative

$X-$Asplund space.

- $f : X \to \mathbb{R} \cup \{+\infty\}$ *Fréchet subdifferential*

$$\partial_F f(x) = \{x^* \in X^* : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\|h\|} \geq 0\}$$

- *Normal cone.* For a closed subset C of X, the normal cone to C at $x \in C$ is defined by $N(C, x) = \partial \delta_C(x)$, where δ_C is the *indicator function* of C given by

  $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise,
- *Coderivative.* Let $F : X \rightrightarrows Y$ be a closed multifunction (graph-closed) and let $(\bar{x}, \bar{y}) \in \text{gph} F$.

  The coderivative of F at $(\bar{x}, \bar{y})$ is the multifunction $D^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$D^* F(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* : (x^*, -y^*) \in N(\text{gph} F, (\bar{x}, \bar{y}))\}.$$

Coderivatives-Examples

- $A \in \mathcal{L}(X, Y)$: $D^*A(y^*) = A^*y^*$.
- $F : X \to Y$ is C^1 : $D^*F(x)(y^*) = (F'(x))^*y^* = y^* \circ F'(x)$.

Estimation for slopes

$X, Y-$ Asplund spaces $\varphi_y(x) = \liminf_{u\to x} d(y, F(u))$.

Theorem. (N.-Tron-Théra - 2011) $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \mathrm{gph}\, F$; *for any subdifferential on* $X \times Y$, *one has*

$$\liminf_{(x,y)\to(\bar{x},\bar{y}), y\notin F(x)} |\nabla \varphi_y|(x) =$$
$$\lim_{\varepsilon\to 0} \inf\{\|x^*\| : \ x^* \in D^*F(u,v)(y^*), \ (u,v) \in B((\bar{x},\bar{y}),\varepsilon), \ \|y^*\| = 1\}.$$

Subdifferential regularity criterion

As a result, (Ioffe 1987)

$$\mathrm{sur}\, F(\bar{x},\bar{y}) = \lim_{\varepsilon\to 0} \inf\{\|x^*\| : \; x^* \in D^*F(u,v)(y^*),\; (u,v)\in B((\bar{x},\bar{y}),\varepsilon),\; \|y^*\| = 1\};$$

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Example

- $f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x) = F(x_1, x_2) := (x_1, x_2^2),\ (x_1, x_2) \in \mathbb{R}^2;$
- $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2,\ F := f - \mathbb{R}^2_+.$
- $f(0,0) + Df(0,0)(\mathbb{R}^2) - \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : \ \ y \le 0\};$
- $0 = (0,0) \notin \mathrm{int}[f(0,0) + Df(0,0)(\mathbb{R}^2) - \mathbb{R}^2_+].$

By (RCQ), F is not metrically regularity at $(0,0)$. However, by directly checking, one has for any $v = (v_1, v_2) \in \mathbb{R}^2$ with $v_2 < 0$,

$$d(x, F^{-1}(y)) \le d(y, F(x)),\ \ \forall (x,y) \text{ near } (0,0),\ y \in F(x) + \text{cone}\{v\}.$$

Remark. *In many practical applications, the metric regularity is too strong, and we need only some weaker regular properties.*

Relative Metric Regularity

Given a subset V of $X \times Y$ and a point $(x, y) \in X \times Y$, we set

$$V_x := \{z \in Y : \; (x, z) \in V\} \quad \text{and} \quad V_y := \{u \in X : \; (u, y) \in V\}.$$

Definition (Ioffe-2010)

Let X and Y be metric spaces, and let $V \subset X \times Y$. We say that a set-valued mapping $F : X \rightrightarrows Y$ is *metrically regular relatively to* V *at* $(\bar{x}, \bar{y}) \in V \cap \mathrm{gph}\; T$ *with a modulus* $\tau > 0$, if there exist $\varepsilon > 0$ such that

$$d\big(x, F^{-1}(y) \cap \mathrm{cl}\, V_y\big) \leq \tau d(y, F(x)) \tag{5}$$

whenever $(x, y) \in \big(B(\bar{x}, \varepsilon) \times B(\bar{y}, \varepsilon)\big) \cap V$ and $d(y, F(x)) < \varepsilon$.

Metric Regularity relative to a cone

$C \subseteq Y$: a nonempty cone;

$$C(\delta) := \{y \in Y : \ \ d(y, C) \leq \delta\|y\|\}, \ \ \delta > 0.$$

- *Metric Regularity relative to C* : F is metrically regular relatively to $V := C(\delta)$, for some $\delta > 0$.
- When $C := \mathrm{cone}\{v\}$: *Directional metric regularity* in v (Arutyunov *et al*-2005).

Relative lower semicontinuous envelop

Given a subset V of $X \times Y$ and a point $(x, y) \in X \times Y$, a multifunction $F : X \rightrightarrows Y$, set

$$V_x := \{z \in Y : \ (x, z) \in V\} \quad \text{and} \quad V_y := \{u \in X : \ (u, y) \in V\}.$$

- *The lower semicontinuous envelop of* $d(y, F(\cdot))$ *relative to* $V \subseteq X \times Y$:

$$\varphi_{F,V}(x, y) := \begin{cases} \liminf\limits_{\mathrm{cl}\, V_y \ni u \to x} d(y, F(u)) & \text{if } x \in \mathrm{cl}\, V_y \\ +\infty & \text{otherwise.} \end{cases} \tag{6}$$

Slope Characterization

Theorem

If There exist $\delta, \gamma > 0$ *such that*

$$|\nabla \varphi_{F,V}(\cdot, y)|(x) \geq \tau^{-1}, \qquad (7)$$

for all $(x, y) \in \big(B(\bar{x}, \delta) \times B(\bar{y}, \delta)\big)$ *with* $\varphi_{F,V}(x, y) \in (0, \gamma)$*, then* F *is metrically regular relative to* V *at* $(\bar{x}, \bar{y})$.

Coderivative characterization

Denote by S_{Y^*} the unit sphere in the continuous dual Y^* of Y, and by d_* the metric associated with the dual norm on X^*. For given $\bar{y} \in Y$ and $\delta > 0$, let us define the set

$$T(C,\delta) := \left\{ \begin{array}{ll} (y_1^*, y_2^*) & \in Y^* \times Y^* : \quad \exists a \in C \cap S_{Y^*}, \\ & \max\{\langle y_1^*, a\rangle, |\langle y_2^*, a\rangle|\} \leq \delta, \ \|y_1^* + y_2^*\| = 1 \end{array} \right\}. \tag{8}$$

- To a given multifunction $F : X \rightrightarrows Y$, we associate the multifunction $G : X \rightrightarrows Y \times Y$ defined by

$$G(x) = F(x) \times F(x), \quad x \in X.$$

- D^*G : *the coderivative of G with respect to the Fréchet subdifferential.*

Theorem

Let X, Y be Asplund spaces and let $F : X \rightrightarrows Y$ be a closed multifunction. Let $(x_0, y_0) \in \mathrm{gph}\, F$ and a nonempty cone $C \subseteq Y$ be given. Assume that F has convex values around x_0, i.e., $F(x)$ is convex for all x near x_0. If

$$\liminf_{\substack{(x,y_1,y_2)\xrightarrow{G}(x_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0, D^*G(x, y_1, y_2)(T(C,\delta))) > m > 0, \tag{9}$$

then F is directionally metrically regular relatively to C with modulus $\tau \le m^{-1}$ at (x_0, y_0). The notation $(x, y_1, y_2) \xrightarrow{G} (x_0, y_0, y_0)$ means that $(x, y_1, y_2) \to (x_0, y_0, y_0)$ with $(x, y_1, y_2) \in \mathrm{gph}\, G$.

Convex multifunctions

Corollary (Ioffe-08)

Let X, Y be Banach spaces and $F : X \rightrightarrows Y$ be a closed convex multifunction and let $(x_0, y_0) \in \operatorname{gph} F$ and $v \in Y$. F is directionally metrically regular in direction v at (x_0, y_0) if and only if

$$\operatorname{cone}\{v\} \cap \operatorname{int}(F(X) - y_0) \neq \emptyset. \tag{10}$$

Robustness

Theorem

Let X be a complete metric space and Y be a normed space. Let $C \subseteq Y$ be a nonempty cone in Y. Let $F : X \rightrightarrows Y$ be a closed multifunction and $(x_0, y_0) \in \operatorname{gph} F$. Suppose that F is metrically regular with a modulus $\tau > 0$ relatively to C. Let $g : X \to Y$ be a mapping locally Lipschitz around x_0 with a Lipschitz constant $L > 0$. Then $F + g$ is metrically regular in the direction $\bar{y}$ at $(x_0, y_0 + g(x_0))$ with modulus

$$\operatorname{reg}_C(F+g)(x_0, y_0 + g(x_0)) \leq \left(\frac{1-\alpha}{\tau(1+\alpha)} - L \right)^{-1},$$

provided

$$\alpha \in (0,1), \quad \textit{and} \quad L < \frac{\delta\alpha}{\tau(1+\alpha)(1+\delta(1-\alpha))}.$$

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Newton's Method

Consider the problem:

$$\text{find } x \text{ such that } \ f(x) = 0,$$

X, Y : Banach spaces; $f : X \to Y$ is of C^1.

Newton's Method for solving this equation is the iterative process:

$$x_{k+1} = x_k - Df(x_k)^{-1} f(x_k),$$

where x_0 is given. $Df(x_k)$ is assumed to be invertible. If the sequence (x_k) converge to ζ such that $Df(\zeta)$ is invertible, then $f(\zeta) = 0$: the non-singular zeros of f correspond to the fixed points of the Newton operator:

$$N_f(x) = x - Df(x)^{-1} f(x).$$

Remark. More generally, when $Df(x)$ is surjectif, the Newton operator is defined by

$$N_f(x) = x - Df(x)^+ f(x),$$

where $Df(x)^+$ denotes the Moore-Penrose generalized inverse ($Df(x)^+ = Df(x)^{-1}$ when $Df(x)$ is invertible)

Quadratic convergence

$U \subseteq X$ is an open set, $f : U \to Y$ is of C^2 on U.

Theorem. *Let $\zeta \in U$ be such that $f(\zeta) = 0$ and let $Df(\zeta)$ be surjectif. For $r > 0$ with $\bar{B}(\zeta, r) \subseteq U$, set*

$$K(f, \zeta, r) = \sup_{\|x-\zeta\| \le r} \|Df(\zeta)^+ D^2 f(x)\|.$$

If $2K(f, \zeta, r)r \le 1$ then for all $x_0 \in \bar{B}(\zeta, r)$, then the Newton sequence $x_{k+1} = N_f(x_k)$ is defined and converges to ζ, quadratically,

$$\|x_k - \zeta\| \le \left(\frac{1}{2}\right)^{2^k - 1} \|x_0 - \zeta\|.$$

Kantorovich theorem

Set $\beta(f, x_0) = \|Df(x_0)^+ f(x_0)\|$ if $Df(x_0)$ is surjectif, and $\beta(f, x_0) = +\infty$, otherwise.

Theorem. (Kantorovich 1949) *Suppose that the following conditions are satisfied:*

- *$Df(x_0)$ is surjectif,*
- $2\beta(f, x_0) \le r$,
- $2\beta(f, x_0)K(f, x_0, r) \le 1$.

then the Newton sequence $x_{k+1} = N_f(x_k)$ is defined and converges to some ζ with $f(\zeta) = 0$, and

$$\|x_k - \zeta\| \le 1.63281... \left(\frac{1}{2}\right)^{2^k - 1} \|x_0 - \zeta\|,$$

with

$$1.63281... = \sum_{k=0}^{\infty} \frac{1}{2^{2^k - 1}}.$$

Newton's method for generalized equations

Problem:

$$\text{find } x \text{ such that } 0 \in f(x) + F(x), \tag{11}$$

where $f : X \to Y$ is a differentiable function and $F : X \rightrightarrows Y$ is a multifunction with closed graph.

Generalized Newton's method for solving (11) : Choose an initial point x_0 and generate a sequence (x_k) iteratively by taking x_{k+1} to be a solution to the auxiliary generalized equation

$$0 \in f(x_k) + Df(x_k)(x - x_k) + F(x) \quad \text{for } k = 0, 1, \ldots \tag{12}$$

Equivalently,

$$x_{k+1} \in (Df(x_k) + F)^{-1}(Df(x_k) - f(x_k)).$$

This method uses *"partial linearization"*: we linearize f at the current point but leave F intact. It reduces to the standard version of Newton's method for solving the nonlinear equation $f(x) = 0$ when $F = 0$.

Quadratic convergence

Theorem (Dontchev-Rockafellar 2009)

Let $\zeta \in X$ be a solution of (11). Consider the Newton method (12) for a continuously differentiable function with $\mathrm{lip}(Df, \zeta) < \infty$. Assume that the mapping $f + F$ is metrically regular at $(\zeta, 0)$. Then for any γ satisfying

$$\gamma > \frac{1}{2}\mathrm{reg}(f + F)(\zeta, 0)\mathrm{lip}(Df, \zeta),$$

there exists a neighborhood $\mathcal{O}$ of ζ such that, for any $x_0 \in \mathcal{O}$, there is a sequence (x_k) generated by the method which converges quadratically to ζ in the sense

$$\|x_{k+1} - \zeta\| \leq \gamma\|x_k - \zeta\|^2, \quad k = 0, 1, ...$$

Quadratic convergence under directional metric regularity

Theorem (N. 2018)

Let $\zeta \in X$ be a solution of (11). Consider Newton's method (12) for a continuously differentiable function with $\mathrm{lip}(Df, \zeta) < \infty$. Assume that the mapping $f + F$ is metrically regular relatively to a cone C at $(\zeta, 0)$. Then there exists a neighborhood $\mathcal{O}$ of ζ such that, for any $x_0 \in \mathcal{O}$, with $0 \in f(x_0) + F(x_0) + C$, there is a sequence (x_k) generated by the method which converges quadratically to ζ.

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THANK YOU FOR YOUR PATIENCE