NON-POLYHEDRAL EXTENSIONS OF THE FRANK AND WOLFE THEOREM

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OUTLINE OF THE PRESENTATION

FW-SETS
cFW SETS
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MOTZKIN FW-SETS
PARABOLIC SETS
q-ASYMPTOTES

FW-SETS

$q : \mathbb{R}^n \to \mathbb{R}$
$q(x) := \frac{1}{2} x^\mathsf{T} A x + b^\mathsf{T} x + c$
$A = A^\mathsf{T} \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$

$F \subseteq \mathbb{R}^n$

F is an *FW*-set if every quadratic function q which is bounded below on F attains its infimum on F.

Every compact set is an *FW*-set.

Every convex polyhedron P is an *FW*-set.
(M. Frank, P. Wolfe, 1956).

PROPOSITION.
FW-sets are closed.

PROPOSITION.
Affine images of FW-sets are FW-sets.

PROPOSITION.
The union of two FW-sets is FW, too.

M affine manifold in $\mathbb{R}^n$
$F \subseteq \mathbb{R}^n$

M is called an *f-asymptote* (Klee, 1960) of F if
$F \cap M = \emptyset$ and dist$(F, M) = 0$.

PROPOSITION.
For a closed convex set $F \subseteq R^n$ and a linear subspace $L \subseteq R^n$, the following statements are equivalent:
1) No translate of L is an f-asymptote of F.
2) The orthogonal projection of F onto $L^\perp$ is closed.
3) $F + L$ is closed.

PROPOSITION.
Let $F \subseteq \mathbb{R}^n$ be an FW-set.
Then F has no f-asymptotes.

PROPOSITION.
Let $F \subseteq \mathbb{R}^n$ be an FW-set and $M \subseteq \mathbb{R}^m$ be an affine manifold.
Then $F \times M$ is an FW-set.

COROLLARY.
Let $F \subseteq \mathbb{R}^n$ be an FW-set and $M \subseteq \mathbb{R}^m$ be an affine manifold.
Then $F + M$ is an FW-set.

PROPOSITION.
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an affine operator and let $F \subseteq T\left(\mathbb{R}^n\right)$ be an FW-set.
Then $T^{-1}(F)$ is an FW-set, too.

cFW-SETS

F is a *qFW*-set if it is convex and the property holds for every quadratic function which is in addition quasiconvex on F.

F is a *cFW*-set if it is convex and the property holds for every quadratic function which is in addition convex on F.

PROPOSITION.
cFW-sets are closed.

PROPOSITION.
Affine images of qFW-sets are qFW-sets.
Affine images of cFW-sets are cFW-sets.

PROPOSITION.

If the union of two qFW-sets is convex, then it is qFW, too.

If the union of two cFW-sets is convex, then it is cFW, too.

THEOREM.

Let $F \subseteq \mathbb{R}^n$ be convex.

Then the following statements are equivalent:

(i) Every polynomial which has at least one nonempty convex sub-level set on F and is bounded below on F attains its infimum on F.

(ii) F is qFW.

(iii) F is cFW.

(iv) F has no f-asymptotes.

(v) $T(F)$ is closed for every affine mapping T.

(vi) $P(F)$ is closed for every orthogonal projection P.

COROLLARY.
A convex set $F \subseteq R^n$ is a cFW-set
if and only if $F + L$
is closed for every linear subspace $L \subseteq R^n$.

PROPOSITION.
If the preimage of a cFW-set under a linear mapping
is nonempty, then it is a cFW-set, too.
PROOF.
If $T^{-1}(F)$ had an f-asymptote M, then $T(M)$ would
be an f-asymptote of F.

EXAMPLE.
$F := \{(x, y) \in \mathbb{R}^2 : x^2 + \exp(-x^2) - y \leq 0\}$
is convex and closed.

F does not have f-asymptotes.

$q(x, y) := y - x^2$

$q(x, y) \geq \exp(-x^2) > 0$ for every $(x, y) \in F$

$$\begin{aligned} 0 &\leq \inf_{(x,y)\in F} q(x, y) \leq \inf_{x\in\mathbb{R}} q\left(x, x^2 + \exp(-x^2)\right) \\ &= \inf_{x\in\mathbb{R}} \exp(-x^2) = 0 \end{aligned}$$

COROLLARY.
Any nonempty finite intersection of cFW-sets is again cFW.

COROLLARY.

Let $F_0 \subseteq \mathbb{R}^n$ be a cFW-set and let $f_1, ..., f_m$ be convex polynomials on F_0 such that

$$F := \{x \in F_0 : f_i(x) \leq 0, i = 1, ..., m\}$$

is nonempty.

Let f be a polynomial which is bounded below on F and has at least one nonempty convex sub-level set on F.

Then f attains its infimum on F.

COROLLARY (Z.-Q. Luo, S. Zhang, 1999).

A nonempty convex region in $\mathbb{R}^n$ defined by convex quadratic constraints is always cFW.

COROLLARY.
If $F_1, ..., F_m$ are cFW-sets,
then $F := F_1 \times ... \times F_m$ is cFW.

PROOF.
Suppose $F_i \subset \mathbb{R}^{d_i}$.
Then

$$\begin{aligned} F \;=\; & \left(F_1 \times \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_m}\right) \\ & \cap \left(\mathbb{R}^{d_1} \times F_2 \times \mathbb{R}^{d_3} \times \cdots \times \mathbb{R}^{d_m}\right) \\ & \cap \cdots \\ & \cap \left(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_{m-1}} \times F_m\right). \end{aligned}$$

EXAMPLE (Z.-Q. Luo, S. Zhang, 1999).

minimize $q(x) := x_1^2 - 2x_1x_2 + x_3x_4$

subject to $c_1(x) := x_1^2 - x_3 \leq 0$

$c_2(x) := x_2^2 - x_4 \leq 0$

$x := (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$F := \{x \in \mathbb{R}^4 : c_1(x) \leq 0, c_2(x) \leq 0\}$ is a cFW-set.

$$\begin{aligned} \inf_{x \in F} q(x) &= \inf \left\{ x_1^2 - 2x_1x_2 + x_1^2x_2^2 \right\} \\ &= \inf \left\{ x_1^2 + (1 - x_1x_2)^2 \right\} - 1 = -1 \end{aligned}$$

$q(x) > -1$ for every $x \in F$

The sum of two FW-sets need not be FW:

$F_1 \times F_2 = F_1 \times \{0\} + \{0\} \times F_2$

MOTZKIN SETS

$F \subseteq \mathbb{R}^n$ closed

F is called Motzkin set (M-set) if there exists a compact set K and a closed convex cone D such that $F = K + D$.

$co\,(F) = co\,(K + D) = co\,(K) + co\,(D) = co\,(K) + D$

$D = 0^+ co(F)$

EXAMPLE.
$D := \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, xy - z^2 \geq 0\}$
is a closed convex cone.

$q(x, y, z) := x^2 + (z - 1)^2$

$q\left(\frac{1}{k}, \frac{(k+1)^2}{k}, 1 + \frac{1}{k}\right) = \frac{2}{k^2} \to 0$
$\inf_{x \in D} q\,(x, y, z) = 0$
$q\,(x, y, z) > 0$ for every $(x, y, z) \in D$

EXAMPLE.

$$F := \left\{ (x, y, z) \in \mathbb{R}^3 : z \geq \left(x^2 + y^2 \right)^{\frac{1}{2}} \right\}$$

is convex and closed.

$$q(x, y, z) := (x - 1)^2 - y + z$$

$$\begin{aligned} (x, y, z) &\in F \\ &\implies \text{either } x \neq 1 \text{ or } z \geq \left(1 + y^2\right)^{\frac{1}{2}} > y \\ &\implies q(x, y, z) > 0 \end{aligned}$$

$$q\left(1, k, \left(1 + k^2\right)^{\frac{1}{2}}\right) = \left(1 + k^2\right)^{\frac{1}{2}} - k \longrightarrow 0$$

THEOREM.
Let $F \subseteq \mathbb{R}^n$ be an M-set.
Then the following statements are equivalent:
(i) F is FW.
(ii) F has no f-asymptotes.
(iii) 0^+F is polyhedral.

PROOF (sketch).

(i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) uses Mirkil's Theorem.

THEOREM (H. Mirkil, 1957).
If a closed convex cone has all its 2-dimensional projections closed, then it is polyhedral.

(iii) $\Longrightarrow$ (i) is based in the following facts:

1) If $F = C + 0^{+}F$, with C compact and convex, and

$$q(x) := \frac{1}{2}x^{\mathsf{T}}Ax + b^{\mathsf{T}}x,$$

then

$$\inf_{x\in F} q(x) = \inf_{y\in C}\left\{q(y) + \inf_{z\in 0^{+}F}\left\{y^{\mathsf{T}}Az + q(z)\right\}\right\}.$$

2) Let D be a polyhedral convex cone and define

$$f(c) := \inf_{x\in D}\left\{c^{\mathsf{T}}x + \frac{1}{2}x^{\mathsf{T}}Gx\right\}.$$

We assume that $x^{\mathsf{T}}Gx \geq 0$ for every $x \in D$.

Then one has:

$$\mathrm{dom}(f) = \left\{c : c^{\mathsf{T}}x \geq 0 \ \forall x \in D \text{ s.t. } x^{\mathsf{T}}Gx = 0\right\}.$$

The right hand side of this equality is a polyhedral convex cone.

Consequently, f is continuous relative to $\mathrm{dom}(f)$.

$$F := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, xy \geq 1\}$$

PROPOSITION.

For a closed convex cone $D \subseteq R^n$ (with $n \geq 2$), the following statements are equivalent:

1) D is polyhedral.

2) $C + D$ is a convex polyhedron for every convex polyhedron C.

3) $L + D$ is closed for every $(n-2)$-dimensional subspace L.

4) D has no $(n-2)$-dimensional f-asymptotes.

MOTZKIN FW-SETS

Motzkin FW-sets:
$F = K + D$
K compact, D polyhedral convex cone

THEOREM.
Any finite intersection of Motzkin FW-sets is a Motzkin FW-set, too.

PROPOSITION.
If the preimage of a Motzkin FW-set F under an affine mapping T is nonempty, it is a Motzkin FW-set, too.
PROOF.

$$T^{-1}(F) = \left(T_{(KerT)^{\perp}}\right)^{-1}(F \cap R(T)) + KerT$$

EXAMPLE.
$F := \left\{(x_1, x_2, x_3) : x_3 \geq \left(x_1^2 + x_2^2\right)^{\frac{1}{2}}\right\}$
$T(x_1, x_2, x_3) := (1, x_2, x_3)$
$x_3 - x_2$ does not attain its infimum on $T^{-1}(F)$.
$T^{-1}(F)$ is not a Motzkin set.

PARABOLIC SETS

THEOREM. (Z.-Q. Luo, S. Zhang, 1999).

Let P be a closed convex polyhedron and define

$$F := \left\{ x \in P : x^T Q x + q^T x + c \leq 0 \right\},$$

where $Q = Q^T \succeq 0$.

Then F is an FW-set.

EXAMPLE.

$F := \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : (x_1 - 1)^2 + x_2^2 \leq 1\right\}$

$F = K + L$

$$K := \left\{(x_1, x_2, 0, 0) \in \mathbb{R}^4 : (x_1 - 1)^2 + x_2^2 \leq 1\right\}$$

$$L := \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}$$

$\mathcal{F} := \left\{x \in F : x_3^2 \leq x_4\right\}$

$q(x_1, x_2, x_3, x_4) := x_1 x_4 - 2x_2 x_3$

$q(x_1, x_2, x_3, x_4) \geq q\left(x_1, x_2, x_3, x_3^2\right) > -2$ for every $(x_1, x_2, x_3, x_4) \in \mathcal{F}$

$q\left(x_1, x_2, x_3, x_3^2\right) \to -2$ for $x_2 := \left(1 - (x_1 - 1)^2\right)^{\frac{1}{2}}$ and $x_3 := \frac{x_2}{x_1}$ as $x_1 \to 0^+$

$\mathcal{F}$ is not an FW-set.

$\mathcal{F}$ is a cFW-set.

$\mathcal{F} = K' \times F'$

$$K' := \left\{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 1)^2 + x_2^2 \leq 1\right\}$$

$$F' := \left\{(x_3, x_4) \in \mathbb{R}^2 : x_3^2 \leq x_4\right\}$$

$\mathcal{F} = F \cap (\{0\} \times \{0\} \times F' + M)$

$$M := \mathbb{R} \times \mathbb{R} \times \{0\} \times \{0\}$$

q-ASYMPTOTES

A nonempty closed set in $\mathbb{R}^n$
F nonempty closed convex set in $\mathbb{R}^n$

A is said to be *asymptotic* to F if
$A \cap F = \emptyset$ and $\text{dist}(F, A) = 0$.

$Q := \{x \in \mathbb{R}^n : q(x) := \frac{1}{2}x^\mathsf{T} Ax + 2b^\mathsf{T} x + c = 0\}$
is a q-*asymptote* of F if
$F \cap Q = \emptyset$ and $\text{dist}(Q \times \{0\}, \{(x, q(x)) : x \in F\}) = 0$.

Q is a q-*asymptote* of F

$$\Longleftrightarrow$$

$Q \times \{0\}$ is asymptotic to $graph\left(q_{|F}\right)$

Q is a q-*asymptote* of $F \Rightarrow Q$ is asymptotic to F

EXAMPLE.
$F := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$
$Q := \{(x, y) \in \mathbb{R}^2 : q(x, y) := xy + 1 = 0\}$

Q is asymptotic to F.

$$q\,(x, y) \geq 1 \text{ for every } (x, y) \in F$$

$$\text{dist}(Q \times \{0\}, \{((x, y)\,, q\,(x, y)) : (x, y) \in F\}) = 1$$

Q is not a q-asymptote of F.

THEOREM.
A convex set F is FW
if and only if
it has no q-asymptotes.

F, Q be closed sets, $F \cap Q = \emptyset$ and $\text{dist}(F, Q) = 0$
Q' closed set.

Q' is *squeezed in between* F and Q if:
$F \cap Q' = \emptyset = Q \cap Q'$
and for every $x \in F$ and $y \in Q$ one has $[x, y] \cap Q' \neq \emptyset$.

$Q_\alpha := \{x \in \mathbb{R}^n : q(x) - \alpha = 0\}$

PROPOSITION.
Let F be a closed convex set.
Then Q_0 is a q-asymptote of F
if and only if
Q_0 is asymptotic to F and no Q_α can be squeezed in between F and Q_0.

PROPOSITION.
Let F be a closed convex set in $\mathbb{R}^n$.
Let $Q := \{x \in \mathbb{R}^n : q(x) = 0\}$ be a quadric.
Suppose Q degenerates to an affine subspace.
Then Q is a q-asymptote of F
if and only if
it is an f-asymptote of F.
Moreover, for any f-asymptote M of F there exists a quadric representation

$$M := \{x \in \mathbb{R}^n : q(x) = 0\},$$

and then M is also a q-asymptote of F.

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