Characterizations of Hölder Error Bounds[1]

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Outline

Outline

Error Bounds

Hoffman (1952); Łojasiewicz (1959);
Robinson (1975); Ioffe (1979); Mangasarian (1985); Auslender, Crouzeix (1988); Burke, Ferris (1993); Cornejo, Jourani, Zălinescu (1997); Pang (1997); Deng (1998); Klatte (1998); Lewis, Pang (1998); Ye (1998); Bauschke, Borwein, Li (1999); Studniarski, Ward (1999); Jourani (2000); Henrion, Outrata (2001, 2005); Ng, Zheng (2001); Azé, Corvellec (2002, 2004, 2014, 2017); Burke, Deng (2002, 2005); Henrion, Jourani (2002); Wu, Ye (2001, 2002, 2003); Azé (2003); Zălinescu (2003); Bosch, Jourani, Henrion (2004); Huang, Ng (2004); Ng, Yang (2004); Corvellec, Motreanu (2008); Ioffe, Outrata (2008); Ngai, Théra (2008, 2009); Penot (2010); Fabian, Henrion, Kruger, Outrata (2010, 2012); Ngai, Kruger, Théra (2010); Bednarczuk, Kruger (2012); Meng, Yang (2012); Chao, Cheng (2014); Kruger (2015, 2016)

Error Bounds

X – metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$, $\bar{x} \in S_f := \{x \in X |\ f(x) \leq 0\}$

Definition

f has a local error bound at $\bar{x}$ if $\exists \tau > 0$, $\delta \in (0, \infty]$ s.t.

$$\tau d(x, S_f) \leq f_+(x) \quad \text{for all } x \in B_\delta(\bar{x})$$

$f_+(x) := \max\{f(x), 0\}$

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- Metric subregularity/calmness of set-valued mappings
- Subtransversality/linear regularity of collections of sets
- Convergence analysis of algorithms
- ...

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$$\mathrm{Er}\, f(\bar{x}) := \liminf_{\substack{x \to \bar{x} \\ f(x) > 0}} \frac{f(x)}{d(x, S_f)} > 0$$

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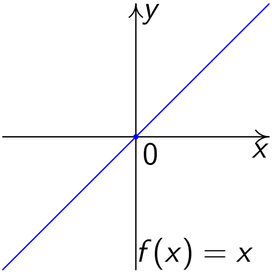

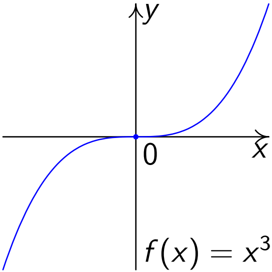

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f has a local error bound of order $q > 0$ at $\bar{x}$ if $\exists \tau > 0$, $\delta \in (0, \infty]$ s.t.

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$$\mathrm{Er}\,{}_q f(\bar{x}) := \liminf_{\substack{x \to \bar{x} \\ f(x) > 0}} \frac{f^q(x)}{d(x, S_f)} > 0$$

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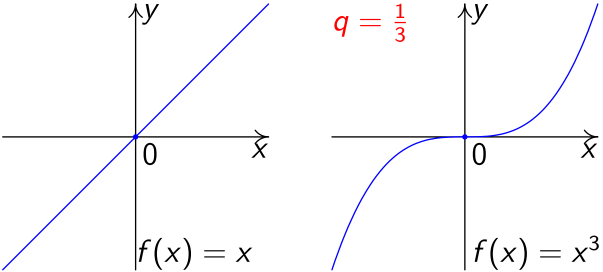

Outline

Error Bounds

X – Asplund space, $f : X \to \mathbb{R} \cup \{+\infty\}$ – lsc, $\bar{x} \in S_f$

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$$\tau d(x, S_f) \le f_+(x) \quad \text{for all } x \in B_\delta(\bar{x})$$

Theorem

If $\exists \delta \in (0, \infty]$ *s.t.* $d(0, \partial f(x)) \ge \tau > 0$ $\forall x \in B_\delta(\bar{x}) \cap [f > 0]$, *then*

$$\tau d(x, S_f) \le f_+(x) \quad \textit{for all} \quad x \in B_{\frac{\delta}{2}}(\bar{x})$$

$\partial f(x)$ – Fréchet subdifferential of f at x

Error Bounds

X – Asplund space, $f : X \to \mathbb{R} \cup \{+\infty\}$ – lsc, $x \notin S_f$, $\tau > 0$

Lemma

Let $0 < f(x) < M$, $\alpha \in (0, 1]$. *If* $d(0, \partial f(u)) \geq \tau$ $\forall u \in X$ *with* $\|u - x\| < \alpha d(x, S_f)$, $f(u) < M$, $f(u) < \tau d(u, S_f)$, *then*

$$\alpha \tau d(x, S_f) \leq f(x)$$

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Tools:

1. Ekeland variational principle
2. (Approximate) sum rule for Fréchet subdifferentials

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$$\mathrm{Er}\, f(\bar{x}) \geq \liminf_{x \to \bar{x},\, f(x) > 0} d(0, \partial f(x))$$

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$S_{f_+^q} = S_f$

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Let $q > 0$. If $\exists \delta \in (0, \infty]$ s.t. $q f^{q-1}(x) d(0, \partial f(x)) \ge \tau > 0$ $\forall x \in B_\delta(\bar{x}) \cap [f > 0]$ with $f^q(x) < \tau d(x, S_f)$, then

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$$\mathrm{Er}\,_q f(\bar{x}) \ge q \liminf_{x \to \bar{x},\, f(x) > 0} \frac{d(0, \partial f(x))}{f^{1-q}(x)}$$

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Theorem (Yao, Zheng, 2016)

Let $q \in (0, 1]$. *If* $\exists \delta \in (0, \infty]$ *s.t.* $d(x, S_f)^{q-1} d(0, \partial f(x))^q \geq \tau$ $\forall x \in B_\delta(\bar{x}) \cap [f > 0]$ *with* $f^q(x) < \tau d(x, S_f)$, *then*

$$\alpha^q (1 - \alpha)^{1-q} \tau d(x, S_f) \leq f_+^q(x) \;\; \textit{for all} \;\; \alpha \in (0, 1) \;\; \textit{and} \;\; x \in B_{\frac{\delta}{1+\alpha}}(\bar{x})$$

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Apply the Lemma with $\tau' := \tau^{\frac{1}{q}}\left((1 - \alpha)d(x, S_f)\right)^{\frac{1}{q}-1}$

Hölder Error Bounds

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or equivalently (with the convention $0^0 = 1$*),*

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Hence, $\quad q^q (1 - q)^{1-q} \tau d(x, S_f) \leq f_+^q(x)$ *for all* $x \in B_{\frac{\delta}{1+q}}(\bar{x})$

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Hence, $\quad q^q (1-q)^{1-q} \tau d(x, S_f) \leq f_+^q(x)$ *for all* $x \in B_{\frac{\delta}{1+q}}(\bar{x})$

$$\mathrm{Er}\,_q f(\bar{x}) \geq q^q (1-q)^{1-q} \liminf_{x \to \bar{x},\, f(x) > 0} \frac{d(0, \partial f(x))^q}{d(x, S_f)^{1-q}}$$

Hölder Error Bounds

X – Asplund space, $f : X \to \mathbb{R} \cup \{+\infty\}$ – lsc, $\bar{x} \in S_f$, $\tau > 0$

Theorem (Quantitative characterizations)

Let $q \in (0, 1]$, $0^0 = 1$. Consider the following conditions:

1. $\tau d(x, S_f) \leq f_+^q(x)$ *for all x near $\bar{x}$*
2. $qf^{q-1}(x)d(0, \partial f(x)) \geq \tau$ *for all $x \in [f > 0]$ near $\bar{x}$*
3. $q^q(1-q)^{1-q}d(x, S_f)^{q-1}d(0, \partial f(x))^q \geq \tau$ *for all $x \in [f > 0]$ near $\bar{x}$*

Then (2) $\Rightarrow$ (1); (3) $\Rightarrow$ (1); (2) $\Rightarrow$ (3) with $(1-q)^{1-q}\tau$ in place of τ
If $q = 1$, then (2) and (3) coincide

Hölder Error Bounds

X – Asplund space, $f : X \to \mathbb{R} \cup \{+\infty\}$ – lsc, $\bar{x} \in S_f$

Corollary (Qualitative characterizations)

Let $q \in (0, 1]$. f has a local error bound of order q at $\bar{x}$ if one of the following conditions is satisfied:

1. $\liminf\limits_{x \to \bar{x},\, f(x) > 0} f^{q-1}(x) d(0, \partial f(x)) > 0$
2. $\liminf\limits_{x \to \bar{x},\, f(x) > 0} d(x, S_f)^{q-1} d(0, \partial f(x))^q > 0$

Outline

Error Bounds: Convex Case

X – Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ – convex lsc, $x \notin S_f$, $\tau > 0$

Lemma

1. *Let* $\alpha \in (0, 1]$. *If* $d(0, \partial f(u)) \geq \tau$ $\forall u \in X$ *with* $\|u - x\| < \alpha d(x, S_f)$, $f(u) \leq f(x)$, $f(u) < \tau d(u, S_f)$, *then* $\alpha \tau d(x, S_f) \leq f(x)$

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2. *If* $\tau d(x, S_f) \leq f(x)$, *then* $d(0, \partial f(x)) \geq \tau$

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2. *If* $\tau d(x, S_f) \leq f(x)$, *then* $d(0, \partial f(x)) \geq \tau$

Theorem

Let $\bar{x} \in S_f$. *The following conditions are equivalent:*

1. $\tau d(x, S_f) \leq f_+(x)$ *for all* x *near* $\bar{x}$
2. $d(0, \partial f(x)) \geq \tau$ *for all* $x \notin S_f$ *near* $\bar{x}$

Hölder Error Bounds: Convex Case

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Lemma

If $q \in (0, 1]$ *and* $\tau d(x, S_f) \leq f^q(x)$, *then*

1. $f^{q-1}(x) d(0, \partial f(x)) \geq \tau$
2. $d(x, S_f)^{q-1} d(0, \partial f(x))^q \geq \tau$

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1. $f^{q-1}(x) d(0, \partial f(x)) \geq \tau$
2. $d(x, S_f)^{q-1} d(0, \partial f(x))^q \geq \tau$

Theorem

Let $\bar{x} \in S_f$. *The following conditions are equivalent:*

1. f *has a local error bound of order* $q \in (0,1]$ *at* $\bar{x}$
2. $\liminf\limits_{x \to \bar{x},\, f(x) > 0} f^{q-1}(x) d(0, \partial f(x)) > 0$
3. $\liminf\limits_{x \to \bar{x},\, f(x) > 0} d(x, S_f)^{q-1} d(0, \partial f(x))^q > 0$

References

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Thank you