Calmness in convex semi-infinite optimization. Modulus estimates

Dedicated to Prof. Alex Kruger

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1. Calmness modulus of the feasible set mapping in linear SIP

2. Hölder calmness of the optimal set in convex SIP

1. Introduction and preliminaries

We consider the parameterized *linear optimization problem*:

$$P(c,a,b): \quad \begin{array}{ll} \text{minimize} & c'x \\ \text{subject to} & a_t'x \leq b_t, \ \ t \in T \end{array}$$

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Symbol $(\bigstar)$ means "this result also holds for *semi-infinite* problems, i.e. when T is infinite".
If T is compact Hausdorff, $t \mapsto a_t \in \mathbb{R}^n$ and $t \mapsto b_t \in \mathbb{R}$ are continuous on T, the problem P is called *continuous*.

CANONICAL *vs* FULL PERTURBATIONS

- **Feasible set mappings,** $\mathcal{F} : \left(\mathbb{R}^{n+1}\right)^T \rightrightarrows \mathbb{R}^n$

$$\mathcal{F}(a,b) := \left\{x \in \mathbb{R}^n : a_t' x \leq b_t, \text{ for all } t \in T\right\},$$

and $\mathcal{F}_{\overline{a}} : \mathbb{R}^T \rightrightarrows \mathbb{R}^n$

$$\mathcal{F}_{\overline{a}}(b) := \mathcal{F}(\overline{a}, b).$$

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- **Optimal set mappings,** $\mathcal{S} : \mathbb{R}^n \times \left(\mathbb{R}^{n+1}\right)^T \rightrightarrows \mathbb{R}^n$,

$$\mathcal{S}(c,a,b) := \left\{x \in \mathbb{R}^n \mid x \text{ is an optimal solution for } P(c,a,b)\right\},$$

and $\mathcal{S}_{\overline{a}} : \mathbb{R}^n \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$

$$\mathcal{S}_{\overline{a}}(c,b) := \mathcal{S}(c,\overline{a},b) \quad (\textit{canonical perturbations}).$$

Topology

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(Recall that $\|u\|_* := \max_{\|x\|\leq 1} |u'x|$).

- For canonical perturbations, when $a = \bar{a}$, $\mathbb{R}^n \times \mathbb{R}^T$ is also endowed with the corresponding supremum norm .

$q-$order error bounds of funtions

Definition

Given the function $f : X \to \mathbb{R} \cup \{+\infty\}$ defined on a metric space X, a point $\bar{x} \in [f \leq 0]$ and a number $q > 0$, we say that f admits a $q-$*order local error bound* at $\bar{x}$, if $\exists\ \kappa \geq 0$ and $\exists$ a neighb. U of $\overline{x}$ such that

$$d(x, [f \leq 0]) \leq \kappa [f(x)]_+^q, \quad \forall x \in U. \tag{1}$$

If $q = 1$, we say that f admits a *local error bound* at $\bar{x}$.

The infimum of all κ in (1) is called the *modulus of q-order error bounds of f at $\bar{x}$*, and it is denoted by $\mathrm{clm_q} f(\bar{x})$.
The absence of q-order error bounds corresponds to $\mathrm{clm_q} f(\bar{x}) = +\infty$.

Hölder calmness of mappings

X, Y metric spaces (distances in X and Y are denoted by d),

Definition

$\mathcal{M} : Y \rightrightarrows X$ is *q-order calm* , $q \in]0,1]$, at $(\bar{y}, \bar{x}) \in \operatorname{gph} \mathcal{M}$ if $\exists U$ neighb. of $\bar{x}$, $\exists V$ neighb. of $\bar{y}$, $\exists \kappa \geq 0$ such that

$$d\left(x, \mathcal{M}\left(\bar{y}\right)\right) \leq \kappa d\left(y, \bar{y}\right)^{q}, \ \forall y \in V, \ \forall x \in \mathcal{M}\left(y\right) \cap U. \qquad (2)$$

$\mathcal{M}$ is *calm* if $q = 1$.
The infimum of all $\kappa \geq 0$ for which (2) holds is called the *q-order calmness modulus* of $\mathcal{M}$ at $(\bar{y}, \bar{x})$; it is

$$\operatorname{clm_q} \mathcal{M}(\bar{y}, \bar{x}) = \limsup_{\substack{y \to \bar{y}, \\ x \to \bar{x}, x \in \mathcal{M}(y)}} \frac{d(x, \mathcal{M}(\bar{y}))}{d(y, \bar{y})^{q}}.$$

If $\operatorname{clm_q} \mathcal{M}(\bar{y}, \bar{x}) = +\infty$, $\mathcal{M}$ is *not* q*−calm* at $(\bar{y}, \bar{x})$.

Calmness under canonical pertubations

Theorem (Robinson, 1981)

If $\mathcal{M} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is polyhedral* ($\operatorname{gph} \mathcal{M}$ *is the finite union of polyhedral sets), then* $\mathcal{M}$ *is calm at any* $(\overline{y}, \overline{x}) \in \operatorname{gph} \mathcal{M}$.

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Corollary

If T is finite, then $\mathcal{F}_{\overline{a}}$ and $\mathcal{S}_{\overline{a}}$ are calm at any element of their graphs.

Remark $\mathcal{S}_{\overline{a}}$ is calm at any point of $\operatorname{gph} \mathcal{S}_{\overline{a}}$ as Karush-Kuhn-Tucker conditions allow us to express the graph of $\mathcal{S}_{\overline{a}}$ as the finite union of polyhedral sets. This is no longer the case for $\mathcal{S}$ in the framework of perturbations of all data.

What about the continuous problem P? For the continuous system $(\overline{a}, \overline{b})$, consider the *supremum function*

$$\overline{s}\,(x) := \max\{\overline{a}_t' x - \overline{b}_t,\ t \in T\}. \tag{3}$$

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For any $x \in \mathbb{R}^n$,

$$\partial \overline{s}\,(x) = \operatorname{conv}\{\overline{a}_t : t \in T_{(\overline{a},\overline{b})}(x)\},$$

where

$$T_{(\overline{a},\overline{b})}(x) := \{t \in T :\ \overline{a}_t' x - \overline{b}_t = \overline{s}\,(x)\}.$$

Due to the continuity of $\overline{s}\,(\cdot)$, $\overline{x} \in \operatorname{bd} \mathcal{F}_{\overline{a}}(\overline{b}) \Rightarrow \overline{s}\,(\overline{x}) = 0$.

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Proposition

$\mathcal{F}_{\overline{a}}$ is calm at $(\overline{b}, \overline{x})$ if and only if $\overline{s}$ has a local error bound at $\overline{x}$; i.e., there exist $\kappa \geq 0$ and a neighb. U of $\overline{x}$ such that

$$d(x, [\overline{s} \leq 0]) \leq \kappa [\overline{s}(x)]_+ , \text{ for all } x \in U.$$

Antecedents

a) *Zheng and Ng'03*: Characterizations of local error bounds in a more general context.
b) *Henrion and Outrata'05*: Criterion for calmness in the context of nonlinear systems under some differentiability assumptions.
c) *Klatte and Kummer'09*: Calmness properties of certain finite nonlinear systems in connection with the convergence of algorithms.
d) *Kruger, Ngai and Théra'10*: Formula for the calmness modulus in a more general context.
e) *Henrion, Jourani and Outrata'02*and *Jourani'00*: Subdifferential approach to calmness/local error bounds.
f) In the 'finite' framework, *Klatte and Thiery'95* and *Li'93* and *Li'94* proved several results about Hoffman constants.

Theorem

If P *is a continuous problem and* $\overline{x} \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b})$, *TFAE:*

(i) $\mathcal{F}_{\overline{a}}$ *is calm at* $(\overline{b}, \overline{x})$

(ii) $\alpha := \liminf_{x \to \overline{x},\ \overline{s}(x) > 0} d_* \left(0_n, \partial \overline{s}\left(x\right)\right) > 0$

(iii) $\beta := \liminf_{x \to \overline{x},\ \overline{s}(x) > 0} \sup_{u \neq x} \frac{\left[\overline{s}\left(x\right) - \left[\overline{s}\left(u\right)\right]_+\right]_+}{d\left(x, u\right)} > 0$

Moreover, we have

$$\mathrm{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) = \alpha^{-1} = \beta^{-1}.$$

Remarks *(i)* $\Leftrightarrow$ *(ii) comes from Azé and Corvellec'04 (Prop. 2.1 and Th. 5.1); (i)* $\Leftrightarrow$ *(iii) from Fabian, Henrion, Kruger and Outrata'12.*

- P satisfies the *Abadie CQ* (ACQ) *around* $\overline{x} \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b})$ if $\exists$ neighb. U of $\overline{x}$ such that

  $$\mathcal{N}(\mathcal{F}_{\overline{a}}(\overline{b}), x) = \overline{\mathrm{cone}}\{\overline{a}_t : t \in T_{(\overline{a},\overline{b})}(x)\} \text{ at any } x \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b}) \cap U,$$

  where cone A is the convex cone generated by A.
- P verifies the *uniform dual boundedness condition* (UDB condition) *around* $\overline{x} \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b})$ if $\exists$ $M > 0$ and a neighb. U of $\overline{x}$ such that

  $$\mathrm{cone}\{\overline{a}_t : t \in T_{(\overline{a},\overline{b})}(x)\} \cap \mathbb{B}_* \subset [0, M]\partial\overline{s}(x), \; \forall x \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b}) \cap U.$$

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Theorem (CLPT'14 Th. 3)

Let $\overline{x} \in \mathrm{bd}\mathcal{F}_{\overline{a}}(\overline{b})$*. Then* $\mathcal{F}_{\overline{a}}$ *is calm at* $(\overline{b}, \overline{x})$ *if and only if* P *satisfies ACQ and UDB around* $\overline{x}$.

Remark This result is inspired in *Zheng and Ng*'03. ACQ and UDB are independent properties.

Calmness modulus of the feasible set mapping

Fix $(\overline{a}, \overline{b}) \in \left(\mathbb{R}^{n+1}\right)^T$, and associated with $\overline{x} \in \mathcal{F}(\overline{a}, \overline{b})$, consider the family of subsets in $T_{(\overline{a},\overline{b})}(\overline{x})$:

$$\mathcal{D}(\overline{x}) := \left\{ D \subset T_{(\overline{a},\overline{b})}(\overline{x}) \;\middle|\; \begin{array}{c} \text{There exists } d \text{ verifying :} \\ \left\{ \begin{array}{ll} \overline{a}_t' d = 1, & t \in D, \\ \overline{a}_t' d < 1, & t \in T(\overline{x}) \setminus D \end{array} \right\} \end{array} \right\}$$

Theorem (CLPT'14, Ths. 4 and 5)

(*i*) *If T is finite,* $\operatorname{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) = \max_{D \in \mathcal{D}(\overline{x})} \left(d_*(0_n, \operatorname{conv}\{\overline{a}_t, t \in D\})\right)^{-1}$

(*ii*) $(\bigstar)$ $\operatorname{clm}\mathcal{F}((\overline{a}, \overline{b}), \overline{x}) = (\|\overline{x}\| + 1)\operatorname{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x})$

Calmness under full perturbations

Theorem (CLPT'14, Cor. 2 (★))

Let $((\overline{a},\overline{b}),\overline{x}) \in \operatorname{gph}\mathcal{F}$; *TFAE:*
(i) $\mathcal{F}$ *is calm at* $((\overline{a},\overline{b}),\overline{x})$;
(ii) $\mathcal{F}_{\overline{a}}$ *is calm at* $(\overline{b},\overline{x})$.

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Theorem (CHaPT'16, Th. 4.1)

Assume that T *is finite and* $\mathcal{S}(\overline{c},\overline{a},\overline{b}) = \{\overline{x}\}$. *The following are equivalent:*
(i) $\mathcal{S}$ *is calm at* $((\overline{c},\overline{a},\overline{b}),\overline{x})$;
(ii) Either Slater holds at $(\overline{a},\overline{b})$ *or* $\mathcal{F}(\overline{a},\overline{b}) = \{\overline{x}\}$;
(iii) $0_n \notin \operatorname{bd}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{(\overline{a},\overline{b})}(\overline{x})\right\}$.

(Slater at $(\overline{a},\overline{b})$: there exists $\widehat{x} \in \mathbb{R}^n$ such that $\overline{a}_t'\widehat{x} < \overline{b}_t$, $t \in T$)

Hölder calmness of the optimal set in convex SIP

Consider the following *convex SIP problem*:

$$\begin{array}{lll} P(c,b): & \text{minimize} & f(x)+c'x \\ & \text{subject to} & g_t(x)\leq b_t,\ t\in T, \end{array}$$

where $c, x \in \mathbb{R}^n$, T is a **compact set**, $f : \mathbb{R}^n \to \mathbb{R}$ and $g_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are **convex functions** such that $(t, x) \mapsto g_t(x)$ is **continuous** on $T \times \mathbb{R}^n$, and $t \mapsto b_t$ is **continuous** on T.

Also now, the pair $(c, b) \in \mathbb{R}^n \times \mathcal{C}(T, \mathbb{R})$ is the parameter to be perturbed, and the parameter space $\mathbb{R}^n \times \mathcal{C}(T, \mathbb{R})$ is endowed with the norm

$$\|(c,b)\| := \max\{\|c\|, \|b\|_\infty\},$$

where **now** $\mathbb{R}^n$ is equipped with the **Euclidean norm** $\|\cdot\|$ and $\|b\|_\infty := \max_{t\in T} |b_t|$.

We deal with the optimal set mapping

$$\mathcal{S} : (c, b) \mapsto \{x \in \mathbb{R}^n \mid x \text{ is optimal for } P(c, b)\},$$

with $(c, b) \in \mathbb{R}^n \times \mathcal{C}(T, \mathbb{R})$.
In the case that $\bar{c}$ is fixed, $\mathcal{S}$ reduces to the partial optimal solution mapping $\mathcal{S}_{\bar{c}} : \mathcal{C}(T, \mathbb{R}) \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{S}_{\bar{c}}(b) = \mathcal{S}(\bar{c}, b).$$

Now, the feasible set mapping is given by

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n \mid g_t(x) \leq b_t, t \in T\},$$

and the set of active indices at $x \in \mathcal{F}(b)$ by

$$T_b(x) := \{t \in T \mid g_t(x) = b_t\}.$$

Definition

The problem $P(c,b)$ satisfies the *Slater constraint qualification* if there exists $\hat{x}$ such that $g_t(\hat{x}) < b_t$ for all $t \in T$.

The following result plays a crucial role in our analysis.

Proposition

Let $(\bar{c},\bar{b}) \in \mathbb{R}^n \times \mathcal{C}(T,\mathbb{R})$ *and assume that* $P(\bar{c},\bar{b})$ *satisfies the Slater condition. Then* $\bar{x} \in \mathcal{S}(\bar{c},\bar{b})$ *if and only if the Karush-Kuhn-Tucker (KKT) conditions hold, i.e.,*

$$\bar{x} \in \mathcal{F}(\bar{b}) \quad \textit{and} \quad -(\partial f(\bar{x}) + \bar{c}) \bigcap \operatorname{cone}\left(\bigcup_{t \in T_{\bar{b}}(\bar{x})} \partial g_t(\bar{x})\right) \neq \emptyset.$$

$\operatorname{cone}(X)$ is the conical convex hull of X; always contains 0_n, entailing $\operatorname{cone}(\emptyset) = \{0_n\}$.

We use the *level set mapping* $\mathcal{L} : \mathbb{R} \times \mathcal{C}(T, \mathbb{R}) \rightrightarrows \mathbb{R}^n$

$$\mathcal{L}(\alpha, b) := \{x \in \mathbb{R}^n \mid f(x) + \bar{c}'x \leq \alpha;\ g_t(x) \leq b_t, t \in T\}$$

and the *supremum function* $\bar{f} : \mathbb{R}^n \to \mathbb{R}$

$$\bar{f}(x) := \sup\{f(x) + \bar{c}'x - \left(f(\bar{x}) + \bar{c}'\bar{x}\right);\ g_t(x) - \bar{b}_t,\ t \in T\}.$$

With $t_0 \notin T$, we define

$$\overline{T} := T \cup \{t_0\},\ g_{t_0}(x) := f(x) + \bar{c}'x \text{ and } \bar{b}_{t_0} := f(\bar{x}) + \bar{c}'\bar{x},$$

and obviously,

$$\bar{f}(x) = \sup\{g_t(x) - \bar{b}_t,\ t \in \overline{T}\}.$$

$\overline{T}$ is compact (as t_0 is an isolated point in $\overline{T}$), the functions $(t, x) \mapsto g_t(x)$ is continuous on $\overline{T} \times \mathbb{R}^n$, $b \in \mathcal{C}(\overline{T}, \mathbb{R})$.

For $x \in \mathbb{R}^n$, the active set is

$$\overline{T}(x) := \{t \in \overline{T} : \bar{f}(x) = g_t(x) - \bar{b}_t\},$$

and

$$\partial \bar{f}(x) = \operatorname{conv}\left(\bigcup_{t \in \overline{T}(x)} \partial g_t(x)\right). \tag{4}$$

Since $((\bar{c}, \bar{b}), \bar{x}) \in \operatorname{gph}(\mathcal{S})$,

$$\mathcal{S}(\bar{c}, \bar{b}) = [\bar{f} = 0] = [\bar{f} \leq 0] = \mathcal{L}(f(\bar{x}) + \langle \bar{c}, \bar{x} \rangle, \bar{b}).$$

Observe that $t_0 \in \overline{T}(\bar{x})$. Consequently $0_n \in \partial \bar{f}(\bar{x})$, and by (4)

$$0_n = \sum_{i=1}^{p} \lambda_i u^i,$$

with $u^i \in \partial g_{t_i}(\bar{x})$, $\{t_i,\ i = 1, 2, \ldots, p\} \subset \overline{T}(\bar{x})$, $\lambda_i > 0$ and $\sum_{i=1}^{p} \lambda_i = 1$.
If $P(\bar{c}, \bar{b})$ satisfies the Slater condition, t_0 must be one of the indices involved in the sum above.

The following lemma provides a uniform boundedness result.

Lemma

Let $((\bar{c},\bar{b}),\bar{x}) \in \mathrm{gph}(\mathcal{S})$ *and assume that* $P(\bar{c},\bar{b})$ *satisfies Slater. Then, there exist* $M>0$ *and neighb.'s* U *of* $\bar{x}$ *and* V *of* $(\bar{c},\bar{b})$ *such that, for all* $(c,b)\in V$ *and all* $x\in\mathcal{S}(c,b)\cap U$*, we have*

$$-(\partial f(x)+c)\bigcap[0,M]\,\mathrm{conv}\left(\bigcup_{t\in T_b(x)}\partial g_t(x)\right)\neq\emptyset. \tag{5}$$

Our approach strongly relies on the following proposition:

Proposition

Let $((\bar{c},\bar{b}),\bar{x}) \in \mathrm{gph}(\mathcal{S})$. *Then the following statements are equivalent:*

(i) $\mathcal{L}$ *is* q *-order calm at* $((f(\bar{x})+\langle \bar{c},\bar{x}\rangle,\bar{b}),\bar{x}) \in \mathrm{gph}(\mathcal{L})$*;*

(ii) $\liminf_{x\to\bar{x},\bar{f}(x)\downarrow 0} \bar{f}(x)^{q-1} d(0,\partial\bar{f}(x)) > 0.$

Moreover,

$$\mathrm{clm_q}\,\mathcal{L}((f(\bar{x})+\langle \bar{c},\bar{x}\rangle,\bar{b}),\bar{x}) = \left(\liminf_{x\to\bar{x},\bar{f}(x)\downarrow 0} \bar{f}(x)^{q-1} d(0,\partial\bar{f}(x))\right)^{-1}.$$

The following theorem constitutes a Hölder convex counterpart of Theorem 3.1 in *Cánovas-Et-Al'14* for the linear case.

Theorem

Let $\bar{x} \in \mathcal{S}(\bar{c}, \bar{b})$ and assume that $P(\bar{c}, \bar{b})$ satisfies the Slater condition. Consider the following statements:

(i) $\mathcal{S}$ is q-order calm at $((\bar{c}, \bar{b}), \bar{x})$;
(ii) $\mathcal{S}_{\bar{c}}$ is q-order calm at $(\bar{b}, \bar{x})$;
(iii) $\mathcal{L}$ is q-order calm at $((f(\bar{x}) + \langle \bar{c}, \bar{x} \rangle, \bar{b}), \bar{x})$;
(iv) $\bar{f}$ has a q-order local error bound at $\bar{x}$.
Then $(iii) \Leftrightarrow (iv) \Rightarrow (i) \Rightarrow (ii)$ hold.

In addition, if f and g_t are linear, then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$.

In the convex setting, $(ii) \Rightarrow (iii)$ could fail:

$$P(0,0): \quad \text{minimize} \quad x^2 \quad \text{subject to} \quad x \leq 0,$$

Take $\bar{c} = 0$, $\bar{b} = 0$, and $\bar{x} = 0$. Then $\mathcal{S}_{\bar{c}}(\bar{b}) = \{0\}$ and $\bar{f}(x) = \sup\{x^2, x\}$.

a) Given $q \in (1/2, 1]$, it is easy to verify that $\liminf_{x \to \bar{x}, \bar{f}(x) \downarrow 0} \bar{f}(x)^{q-1} d(0, \partial\bar{f}(x)) = 0$ and, by Proposition 3, $\mathcal{L}$ is not q-order calm at $((0,0),0) \in \operatorname{gph}(\mathcal{L})$.

b) On the other hand, we have

$$\mathcal{S}_{\bar{c}}(b) = \min\{0, b\}, \text{ for all } b \in (-1,1). \tag{6}$$

Since $\|b\| \leq \|b\|^{\frac{2}{3}}$ $\forall b \in (-1,1)$, it follows from (6)

$$d(x, \mathcal{S}_{\bar{c}}(\bar{b})) \leq \|b - \bar{b}\|^{\frac{2}{3}} \quad \forall x \in \mathcal{S}_{\bar{c}}(b) \cap (-1,1) \text{ and } b \in (-1,1), \tag{7}$$

i,e., $\mathcal{S}_{\bar{c}}$ is $2/3$-order calm at $(0,0)$.

Definition

The *Extended Nürnberger Condition* (*ENC*) is satisfied at $((\bar{c},\bar{b}),\bar{x}) \in \mathrm{gph}(\mathcal{S})$ if

$$P(\bar{c},\bar{b}) \text{ satisfies Slater and } \nexists\, D \subset T_{\bar{b}}(\bar{x}) \text{ with } |D| < n$$

$$\text{such that } -(\partial f(\bar{x}) + \bar{c}) \bigcap \mathrm{cone}\left(\bigcup_{t\in D} \partial g_t(\bar{x})\right) \neq \emptyset.$$

The parameter c can be fixed when ENC is fulfilled!

Theorem

Let $((\bar{c},\bar{b}),\bar{x}) \in \mathrm{gph}(\mathcal{S})$ *and suppose that ENC is satisfied at* $((\bar{c},\bar{b}),\bar{x})$*. Then*

$$\mathrm{clm_q}\,\mathcal{S}((\bar{c},\bar{b}),\bar{x}) = \mathrm{clm_q}\,\mathcal{S}_{\bar{c}}(\bar{b},\bar{x}).$$

Next we consider a weaker condition yielding a lower estimate for $\mathrm{clm_q}\mathcal{S}((\bar{c},\bar{b}),\bar{x})$.
We associate with $(b,x)\in \mathrm{gph}\,\mathcal{S}_{\bar{c}}$ the family of KKT index sets

$$\mathcal{M}_b(x) := \left\{ D\subset T_b(x) \;\middle|\; \begin{array}{c} -(\partial f(x)+c)\bigcap \mathrm{cone}\left(\bigcup_{t\in D}\partial g_t(x)\right)\neq\emptyset \\ \text{and } D \text{ is minimal for the inclusion order} \end{array} \right\}$$

To any $D\in\mathcal{M}_{\bar{b}}(\bar{x})$, we associate the function $f_D:\mathbb{R}^n\to\mathbb{R}$

$$f_D(x) := \sup\{g_t(x)-\bar{b}_t, t\in T;\ -g_t(x)+\bar{b}_t, t\in D\}.$$

Theorem

Let $\mathcal{S}(\bar{c},\bar{b})=\{\bar{x}\}$ and assume that $P(\bar{c},\bar{b})$ satisfies Slater. Then

$$\mathrm{clm_q}\,\mathcal{S}((\bar{c},\bar{b}),\bar{x}) \geq \left(\inf_{D\in\mathcal{M}_{\bar{b}}(\bar{x})} \liminf_{\substack{x\to\bar{x}\\ f_D(x)>0}} f_D(x)^{q-1} d(0,\partial f_D(x)) \right)^{-1}.$$

Finally, we will consider the linear counterpart of $P(c, b)$; namely, we will always assume that $f = 0$ and $g_t(x) = a_t' x$ for all $t \in T$ therein, where $t \mapsto a_t \in \mathbb{R}^n$ is continuous on T.

Proposition

Let $\mathcal{S}(\bar{c}, \bar{b}) = \{\bar{x}\}$ and assume that $P(\bar{c}, \bar{b})$ satisfies the Slater condition. Then the following estimates hold

$$\begin{aligned} \operatorname{clm_q} \mathcal{S}((\bar{c}, \bar{b}), \bar{x}) &\geq \operatorname{clm_q} \mathcal{S}_{\bar{c}}(\bar{b}, \bar{x}) \\ &\geq \left(\inf_{D \in \mathcal{M}_{\bar{b}}(\bar{x})} \liminf_{\substack{x \to \bar{x} \\ f_D(x) > 0}} f_D(x)^{q-1} d(0, \partial f_D(x)) \right)^{-1} \\ &= \sup_{D \in \mathcal{M}_{\bar{b}}(\bar{x})} \left(\operatorname{clm_q} f_D(\bar{x}) \right). \end{aligned}$$

Also in the linear programming setting, and with $q = 1$:

Theorem

(*i*) ***(CHeLP'16, Cor. 4.1)***

$$\operatorname{clm} \mathcal{S}((\bar{c}, \bar{b}), \bar{x}) = \sup_{D \in \mathcal{M}_{\bar{b}}(\bar{x})} (\operatorname{clm} f_D(\bar{x})) .$$

(*ii*) ***(CHePT'16, §5)*** *Assume that Slater holds and* $\mathcal{S}(\bar{c},\bar{a},\bar{b}) = \{\bar{x}\}$. *Then*

$$\operatorname{clm} \mathcal{S}((\bar{c},\bar{a},\bar{b}),\bar{x}) = (\|\bar{x}\| + 1) \operatorname{clm} \mathcal{S}_{\bar{a}}((\bar{c},\bar{b}),\bar{x}),$$

if $\|\bar{c}\|_*$ *is small enough (critical objective size).*

First results on the calmness of $\mathcal{S}$, under uniquenes of optimal solution:

- Cánovas, Kruger, López, Parra, Théra, *SIOPT*, 2014
- Cánovas, Hantoute, Parra, Toledo, *Math. Program.*, 2016

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More results:

- Cánovas, Henrion, López, Parra, *JOTA*, 2016
- Cánovas, Henrion, Parra, Toledo, *Set-Valued V. A.*, 2016

Applications

On the convergence of certain algorithms

- Cánovas, Hantoute, Parra, Toledo, Math. Program. (2015).
  - A descent method (by Klatte and Kummer) in LP
  - A regularization method (by Kadrani, Dussault and Benchakroun) for linear MPCC's
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- Cánovas, Hall, López, Parra, Optimization, 2018
  - Interior point methods

Application in robust optimization

- Cánovas, Henrion, López, Parra, Stud. Syst. Decis. Control, 142, Springer 2018
  - Calmness constants for uncertain linear inequality systems

A uniform approach to Hölder calmness of subdifferentials

- Consider the set-valued mapping $\mathcal{S} : \Gamma \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by
$$\mathcal{S}(f, x) := \partial f(x),$$
where Γ represents the family of all finite-valued convex functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

A uniform approach to Hölder calmness of subdifferentials

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where Γ represents the family of all finite-valued convex functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

- Given $x_0 \in \mathbb{R}^n$, our aim is to quantify the *stability* of $\mathcal{S}$ around x_0 and *uniformly* with respect to f; i.e. involving pairs of functions f_1 and f_2, close enough to each other (with respect to the standard uniformity for the topology of uniform convergence on bounded subsets).

Next we present some preliminary results.

Proposition

(Upper semicontinuity of $\mathcal{S}$*) Let* $(f_0, x_0) \in \Gamma \times \mathbb{R}^n$*. Given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\partial f(x) \subset \partial f_0(x_0) + \varepsilon \mathbb{B};$$

provided that $f \in \Gamma$ *satisfies*

$$d_\alpha(f, f_0) := \sup_{z \in x_0 + \alpha \mathbb{B}} |f(z) - f_0(z)| \leq \delta,$$

and $\|x - x_0\| \leq \delta$.

The metric of the uniform convergence on compact subsets of $\mathbb{R}^n$ is

$$\rho(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \min\left\{d_k(f,g), 1\right\}, \tag{8}$$

where

$$d_k(f,g) = \max\{|f(x) - g(x)| : \|x - x_0\| \le k\}$$

and x_0 is a fixed point.
Metric ρ is not really helpful, and it is easier to work with d_α.
Moreover, for each $\varepsilon > 0$, there exist $k \in \mathbb{N}$ and $\delta > 0$ such that $\rho(f,g) < \varepsilon$ for each pair of functions (f,g) satisfying $d_k(f,g) < \delta$.

The following result follows from Cor. 4.3 in *Aragón-Geoffroy'14* and Fenchel's equality.

Proposition

Let $(f_0, x_0) \in \Gamma \times \mathbb{R}^n$ *and* $u_0 \in \partial f_0(x_0)$ *be given. Then* ∂f_0 *is calm at* (x_0, u_0) *if and only if there exist a neighb.* U *of* u_0 *and a positive constant* c *such that*

$$f_0^*(u) + f_0(x_0) \geq u' x_0 + c d(u, \partial f(x_0))^2 \text{ for all } u \in U. \tag{9}$$

Specifically, if ∂f_0 *is calm at* (x_0, u_0) *with constant* κ*, then (9) holds for all* $c < 1/(4\kappa)$*; conversely, if (9) holds with constant* c*, then* ∂f *is calm at* (x_0, u_0) *with constant* $1/c$*.*

The next result is Theorem 5.1 in *Beer-Cánovas-L-Parra'2018*:

Theorem

Let $(f_0, x_0) \in \Gamma \times \mathbb{R}^n$ *and fix* $\alpha > 0$. *Assume that* ∂f_0 *is calm at* (x_0, u) *for any* $u \in \partial f_0(x_0)$. *Then, there exist* $\kappa > 0$ *and* $0 < \delta_0 \leq 1$ *such that for any* $(f, x) \in \Gamma \times \mathbb{R}^n$ *verifying*

$$d((f,x),(f_0,x_0)) := \max\{d_\alpha(f,f_0), \|x - x_0\|\} \leq \delta_0$$

we have

$$d(u, \mathcal{S}(f_0,x_0)) \leq \kappa_1 \sqrt{d((f,x),(f_0,x_0))} \text{ for all } u \in \mathcal{S}(f,x).$$

In other words, $\mathcal{S}$ *is* $(1/2)$*-Hölder calm at* (f_0, x_0).

Here we use a pseudometric in the image space.

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Antecedents

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