Extremal Principle and its extension

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Centre for Informatics and Applied Optimization, Federation University Australia
WoMBaT workshop

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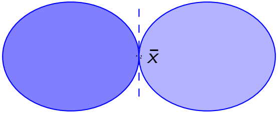
Classical separation theorem

X–normed linear vector space



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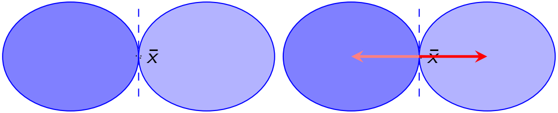

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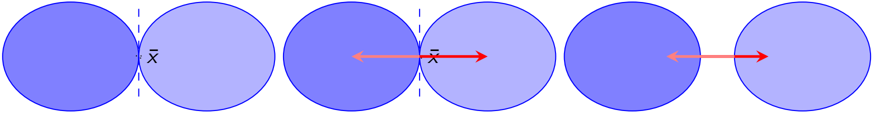

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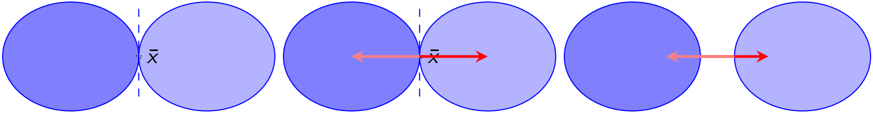
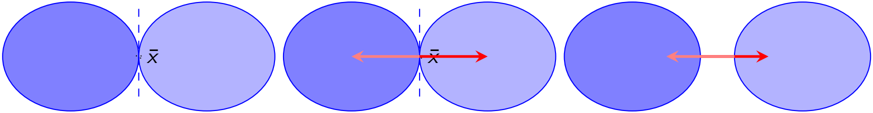


$C_1,\ C_2 \neq \emptyset$–convex sets. We can separate C_1, C_2 if there exists $x^* \in X^*, x^* \neq 0$ such that

$$\inf_{x \in C_1} \langle x^*, x\rangle \geq \sup_{x \in C_2} \langle x^*, x\rangle.$$

X–normed linear vector space



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Theorem (M. Fabian, N. V. Živkov, 1985)

C_1, C_2 can be separated if and only if there exists a cone K with non-empty interior such that

$$(C_1 - C_2) \cap K \subset \{0\}.$$

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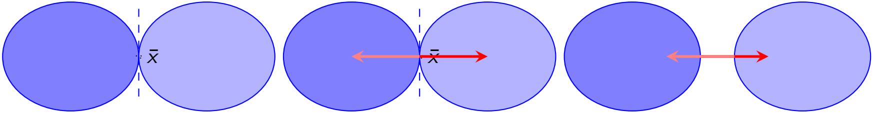

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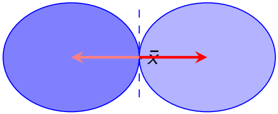
$\operatorname{int} C_1 \neq \emptyset$, $\bar{x} \in C_1 \cap C_2$.

Theorem (Separation Theorem)

If $\operatorname{int} C_1 \cap C_2 = \emptyset$, then $\exists x^* \in X^* \setminus \{0\}$ s.t. $x^* \in N_{C_1}(\bar{x})$, $-x^* \in N_{C_2}(\bar{x})$.

$N_C(\bar{x}) := \{x^* \in X : \langle x^*, x - \bar{x} \rangle \leq 0, \forall x \in C\}.$

Non–convex setting



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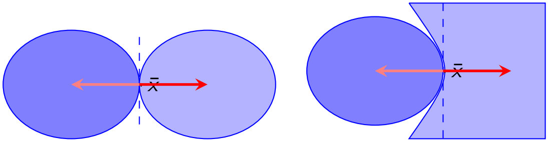

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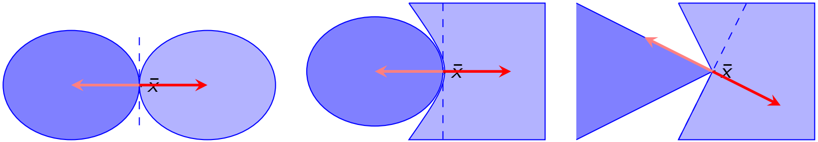

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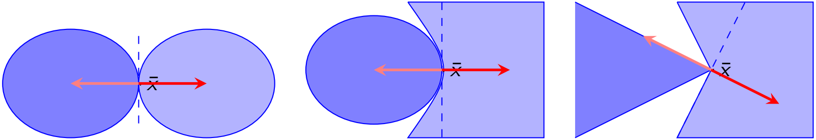

$0 \notin \text{ int } (A - B)$.

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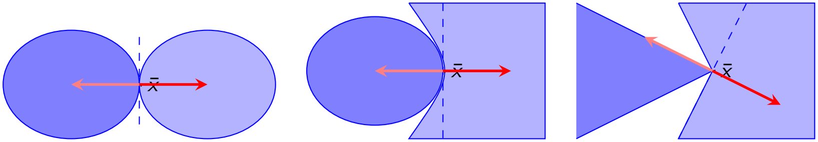

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Definition

The pair $\{A, B\}$ is

1. extremal if $\forall \varepsilon > 0$, $\exists u, v \in X$ such that
$$(A - u) \cap (B - v) = \emptyset \quad \text{and} \quad \max\{\|u\|, \|v\|\} < \varepsilon;$$
2. locally extremal at $\bar{x} \in A \cap B$ if $\exists \rho > 0$ such that $\forall \varepsilon > 0$, $\exists u, v \in X$ satisfying
$$(A - u) \cap (B - v) \cap \mathbb{B}_\rho(\bar{x}) = \emptyset \quad \text{and} \quad \max\{\|u\|, \|v\|\} < \varepsilon.$$

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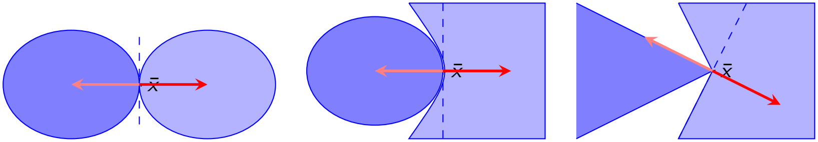

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Theorem

$\dim X < +\infty$, A, B–closed, if $\{A, B\}$ is locally extremal at $\bar{x} \in A \cap B$, then

$$\bar{N}_A(\bar{x}) \cap (-\bar{N}_B(\bar{x})) \neq \{0\}.$$

Normal cone?

X–normed vector space, $A \in X$, $\bar{x} \in A$

Definition

Fréchet normal cone to A at $\bar{x}$

$$N_A(\bar{x}) := \left\{ x^* \in X^* : \limsup_{x \to \bar{x}, x \in A \setminus \{\bar{x}\}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}.$$

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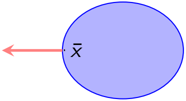

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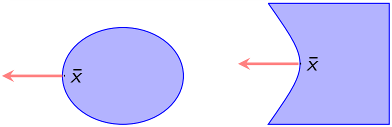

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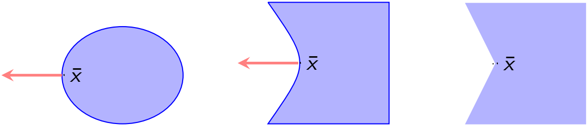

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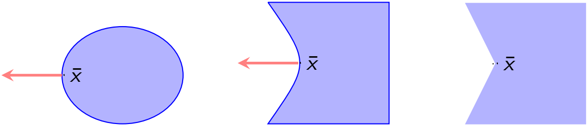

Definition

$\dim X < +\infty, A \subset X, \bar{x} \in A$, the limiting normal cone to A at $\bar{x}$

$$\bar{N}_A(\bar{x}) := \limsup_{x \to \bar{x}, x \in A \setminus \{\bar{x}\}} N_A(x).$$

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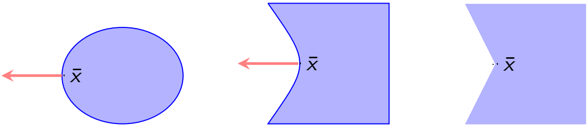

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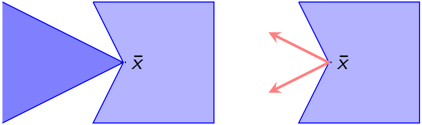

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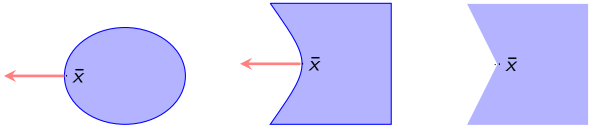

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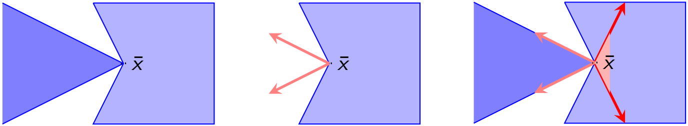

X–Asplund space, $A, B \subset X$ are closed, and $\bar{x} \in A \cap B$.

Theorem (Extremal principle)

If the pair $\{A, B\}$ is locally extremal at $\bar{x}$, then for any $\varepsilon > 0$ there exist points $a \in A \cap B_\varepsilon(\bar{x})$, $b \in B \cap B_\varepsilon(\bar{x})$, $a^* \in N_A(a)$ and $b^* \in N_B(b)$ such that

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$\varepsilon > 0$ there exist points $a \in A \cap B_\varepsilon(\bar{x})$, $b \in B \cap B_\varepsilon(\bar{x})$, $x^* \in X^*$ s.t.

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Corollary (Density of 'support' points)

$A \subset X$, A is closed, $\bar{x} \in \mathrm{bd}\, A$. Then for all $\varepsilon > 0$, there is $x \in \mathbb{B}_\varepsilon(\bar{x})$ such that $N_A(x) \neq \{0\}$.

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A, $B := \{\bar{x}\}$ are extremal.

Corollary ('exact' version)

$A - B$–closed, $\{A, B\}$–extremal, then $\forall \varepsilon > 0$, $\exists a \in A$, $b \in B$, $x^* \in X^*, x^* \neq 0$ s.t.

$$\|a - b\| < \varepsilon, \quad x^* \in N_A(a), \quad -x^* \in N_B(b).$$

Definition

The pair $\{A, B\}$ is approximately stationary at $\bar{x}$ if $\forall \varepsilon > 0$, $\exists \rho \in (0, \varepsilon)$, $a \in A \cap \mathbb{B}_\varepsilon(\bar{x})$, $b \in B \cap \mathbb{B}_\varepsilon(\bar{x})$ and $u, v \in X$ such that

$$(A - a - u) \cap (B - b - v) \cap (\rho\mathbb{B}) = \emptyset \quad \text{and} \quad \max\{\|u\|, \|v\|\} < \varepsilon\rho.$$

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Definition

The pair $\{A, B\}$ is transversal at $\bar{x}$ if $\exists\alpha, \delta > 0$ such that

$$\alpha d(x, (A - x_1) \cap (B - x_2)) \le \max\{d(x, A - x_1), d(x, B - x_2)\}$$

for all $x \in \mathbb{B}_\delta(\bar{x}), x_1, x_2 \in (\delta\mathbb{B})$.

Approximate stationarity & Transversality

Definition

The pair $\{A, B\}$ is approximately stationary at $\bar{x}$ if $\forall \varepsilon > 0$, $\exists \rho \in (0, \varepsilon)$, $a \in A \cap \mathbb{B}_\varepsilon(\bar{x})$, $b \in B \cap \mathbb{B}_\varepsilon(\bar{x})$ and $u, v \in X$ such that

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for all $x \in \mathbb{B}_\delta(\bar{x}), x_1, x_2 \in (\delta \mathbb{B})$.

Theorem (A. Kruger, 1998, 2002)

If the pair $\{A, B\}$ is approximately stationary at $\bar{x}$, if and only if for any $\varepsilon > 0$ there exist points $a \in A \cap B_\varepsilon(\bar{x})$, $b \in B \cap B_\varepsilon(\bar{x})$, $a^* \in N_A(a)$ and $b^* \in N_B(b)$ such that

$$\|a^*\| + \|b^*\| = 1 \quad \textit{and} \quad \|a^* + b^*\| < \varepsilon.$$

Recent Extensions

X–Asplund, A, B–closed, $\bar{x} \in A \cap B$

Theorem (A. Kruger, M. Lôpez, 2012)

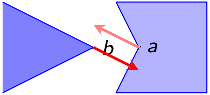
1. *If* $\varepsilon > 0$, $a \in A \cap B_\varepsilon(\bar{x})$, $b \in B_\varepsilon(\bar{x})$, $\rho > 0$, $u, v \in (\rho\varepsilon\mathbb{B})$ *s.t.*

$$(A - a - u) \cap (B - b - v) \cap (\rho\mathbb{B}) = \emptyset, \quad (1)$$

then, for all $\delta > \varepsilon + \rho(\varepsilon + 1)$ *there are* $x \in B_\delta(\bar{x})$, $y \in B_\delta(\bar{x})$, $x^* \in N_A(x)$, $y^* \in N_B(y)$ *s.t*

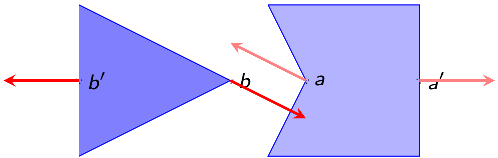
$$\|x^*\| + \|y^*\| = 1, \quad \|x^* + y^*\| < \varepsilon. \quad (2)$$

2. *If* $\varepsilon > 0$, $x \in A$, $y \in B$, $y^* \in N_A(x)$, $y^* \in N_B(y)$ *s.t.* (2) *holds, then* $\forall\delta > 0, \exists\rho \in (0, \delta)$ *s.t.* (1) *holds.*

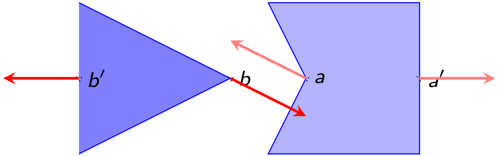


Theorem (H. Bui, A. Kruger, 2017)

$\{A, B\}$ *is relative approximately stationary at* (a, b) *if and only if* $\forall\varepsilon > 0$, $\exists x \in A \cap B_\varepsilon(a)$, $y \in B \cap B_\varepsilon(b)$, $x^* \in N_A(x)$, $y^* \in N_B(x)$ *s.t.* (2) *holds.*



b′
b
a
a′



Theorem (Zheng&Ng, 2011)

X–Asplund, A, B–closed, $A \cap B = \emptyset$, $a \in A$, $b \in B$, $\varepsilon > 0$

$$\|a - b\| < d(A, B) + \varepsilon,$$

then, for all $\lambda > 0$, $\tau \in (0, 1)$, $\exists x \in A \cap B_\lambda(a)$, $y \in B \cap B_\lambda(b)$, $x^* \in X^*$ s.t.

$$\|x^*\| = 1, \quad d(x^*, N_A(x)) + d(x^*, -N_B(y)) < \varepsilon/\lambda,$$
$$\langle x^*, x - y\rangle > \tau \left\|a' - b'\right\|.$$

An Extension

X–Asplund, A, B–closed, $\bar{x} \in A \cap B$,

Theorem

If $\varepsilon > 0$, $u, v \in X$ *s.t.*

$$(A - u) \cap (B - v) = \emptyset,$$
$$\|u\|, \|v\| < d(A - u, B - v) + \varepsilon$$

(or simply $\|u\|, \|v\| < \varepsilon$*). Then, for any* $\tau \in (0, 1)$, $\lambda > 0$ *and* $\rho > 0$*, there exist points* $a \in A \cap B_\lambda(\bar{x})$, $b \in B \cap B_\lambda(\bar{x})$, $x \in B_\rho(\bar{x})$, *and vectors* $a^*, b^* \in X^*$ *s.t.*

$$\|a^*\| + \|b^*\| = 1,$$
$$\lambda\left(d(a^*, N_A(a)) + d(b^*, N_B(b)\right) + \rho\,\|a^* + b^*\| < \varepsilon,$$
$$\langle a^*, x + u - a\rangle + \langle b^*, x + v - b\rangle > \tau \max\{\|x + u - a\|, \|x + v - b\|\}$$

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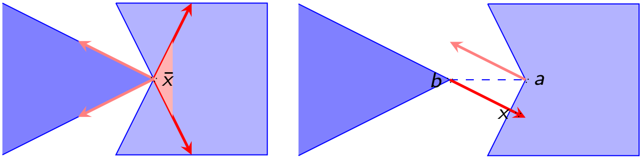
$$(A - u) \cap (B - v) = \emptyset,$$
$$\|u\|, \|v\| < d(A - u, B - v) + \varepsilon$$

(or simply $\|u\|, \|v\| < \varepsilon$*). Then, for any* $\tau \in (0,1)$, $\lambda > 0$ *and* $\rho > 0$, *there exist points* $a \in A \cap B_\lambda(\bar{x})$, $b \in B \cap B_\lambda(\bar{x})$, $x \in B_\rho(\bar{x})$, *and vectors* $a^*, b^* \in X^*$ *s.t.*

$$\|a^*\| + \|b^*\| = 1,$$
$$\lambda \left(d(a^*, N_A(a)) + d(b^*, N_B(b)\right) + \rho \|a^* + b^*\| < \varepsilon,$$
$$\langle a^*, x + u - a\rangle + \langle b^*, x + v - b\rangle > \tau \max\{\|x + u - a\|, \|x + v - b\|\}$$



Thank You!