# On higher order Voronoi cells

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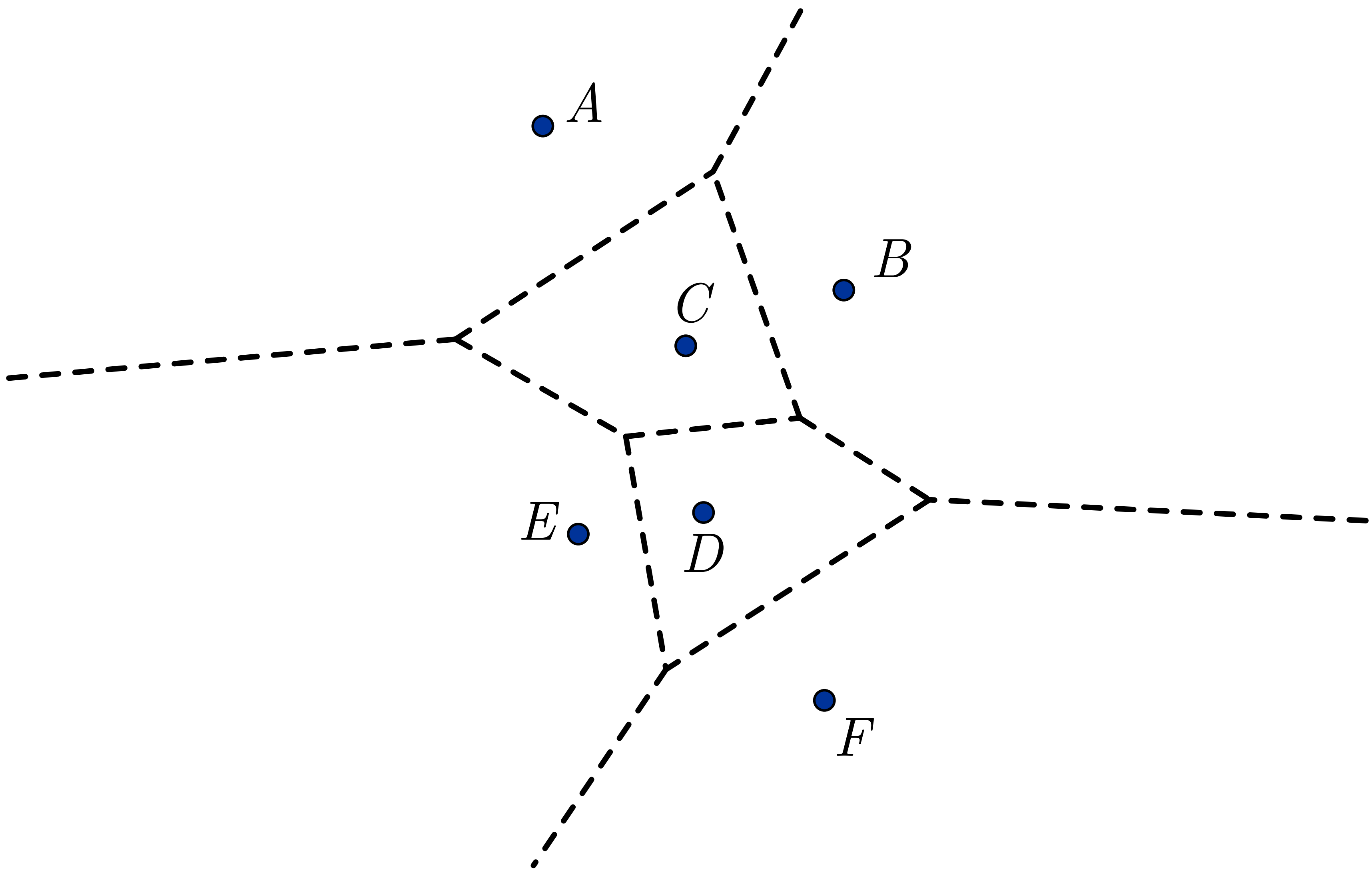
# Voronoi cells



# Basic facts about Voronoi diagrams

Let $T \subset \mathbb{R}^n$ be a finite set. The Voronoi cell of $s \in T$ is

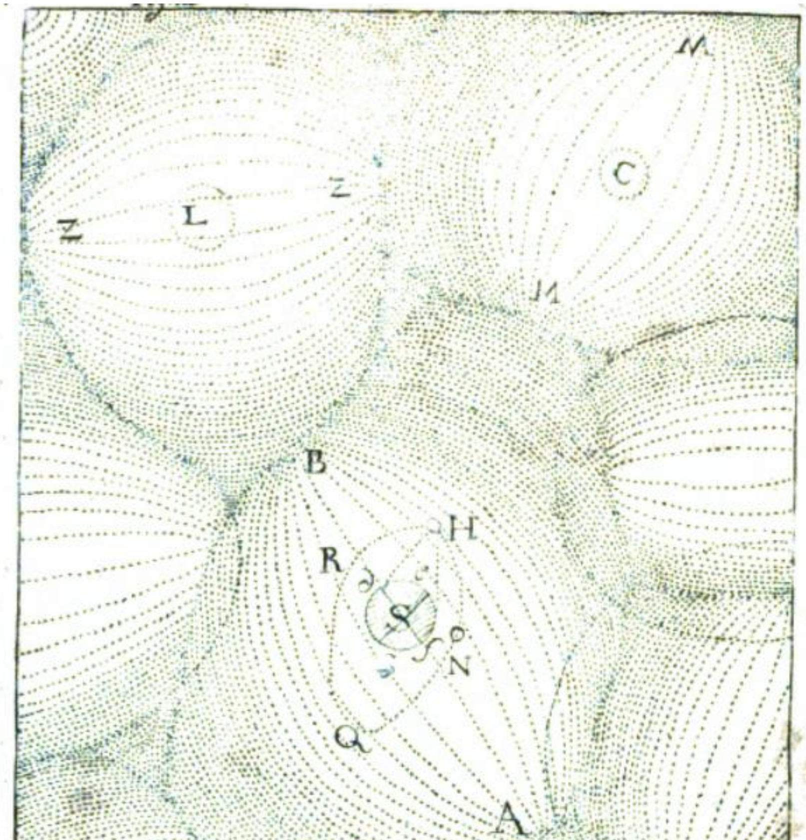
$$V_T(s) = \left\{ x \in \mathbb{R}^n \,:\, \text{dist}(x,s) \le \min_{t \in T \setminus \{s\}} \text{dist}(x,t) \right\}.$$

Voronoi cells are used in computer graphics, crystallography, facility location, and other areas including school catchment areas.

First recorded use in [René Descartes, Principia philosophiae, 1644].

[G.L. Dirichlet. Über die reduktion der positiven quadratischen formen mid drei unbestimmten ganzen zahlen, 1850]

[M.G. Voronoi. Nouvelles applications des paramètres continus à la théorie des formes quadratiques, 1908]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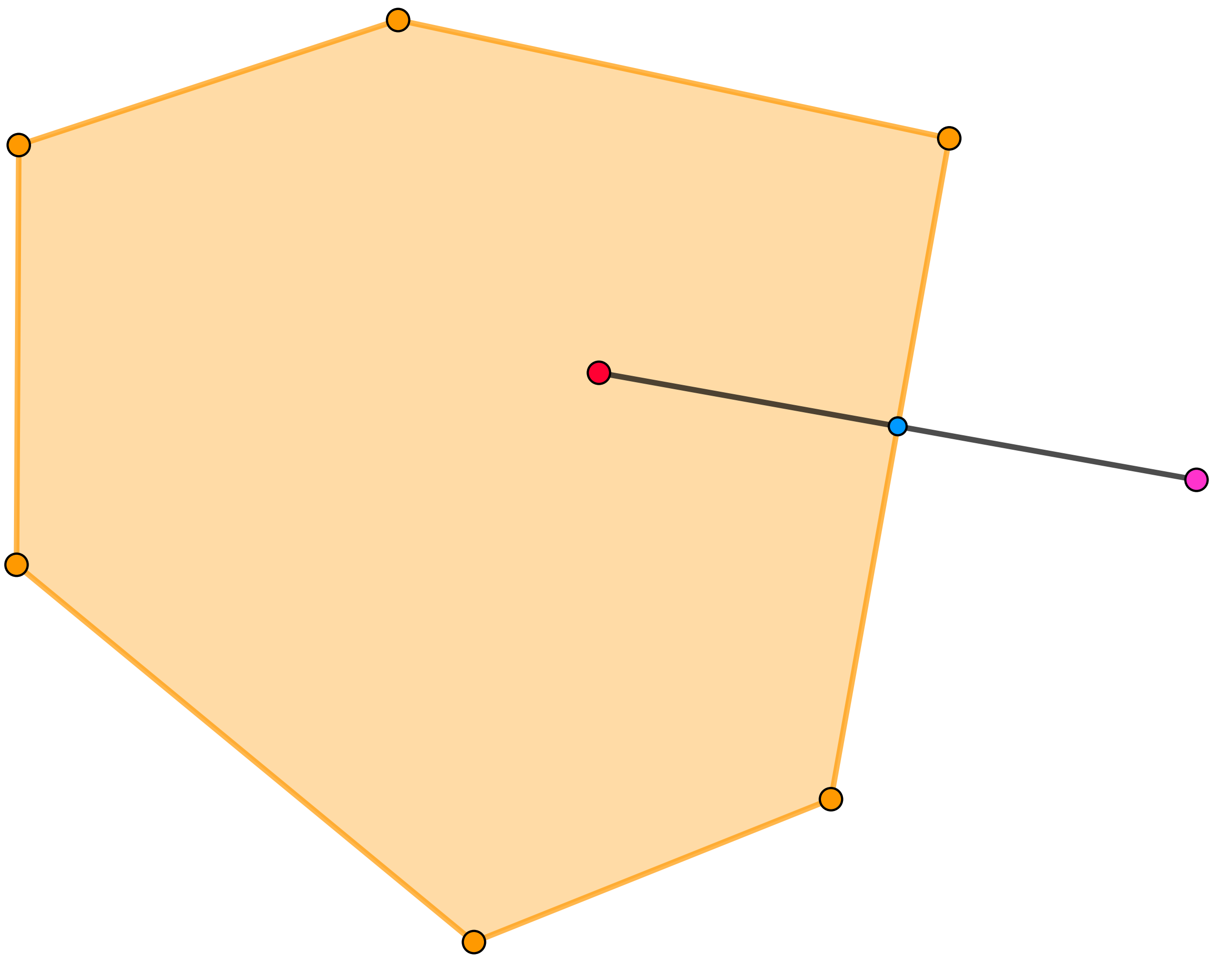
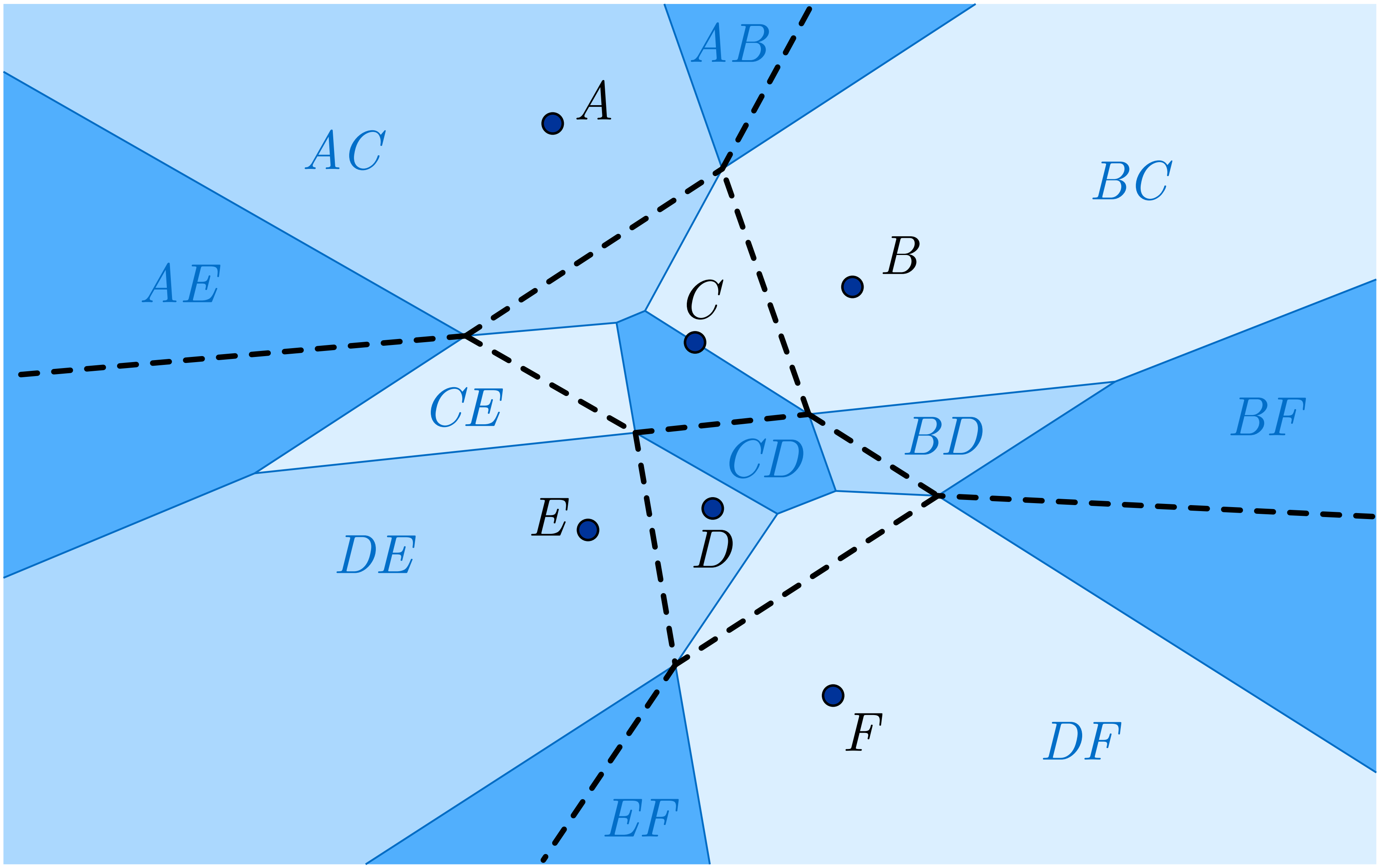
# Every polyhedral set is a Voronoi cell



# Higher order (multipoint) Voronoi cells



# Higher order (multipoint) Voronoi cells

Higher order cells are utilized in a numerical technique for smoothing point clouds from experimental data, for detecting and rectifying coverage problems in wireless sensor networks, to analyze coalitions in the US supreme court voting decisions and a $k$-nearest neighbor problem in spatial networks.

# Higher order cells are polyhedral

$$V_T(S) := \left\{ x \in \mathbb{R}^n \,:\, \max_{s \in S} \operatorname{dist}(x, s) \leq \min_{t \in T \setminus S} \operatorname{dist}(x, t) \right\}$$

**Proposition 1.** *Let $T$ be a finite subset of $\mathbb{R}^n$, and let $S$ be a nonempty proper subset of $T$. Then $V_T(S)$ is the intersection of $|S|(|T| - |S|)$ closed halfspaces:*

$$V_T(S) = \bigcap_{\substack{s \in S \\ t \in T \setminus S}} \left\{ x \in \mathbb{R}^n \,:\, \langle t - s, x \rangle \leq \frac{1}{2} \left( \|t\|^2 - \|s\|^2 \right) \right\}.$$

# Empty cells

**Theorem 2.** *Let $T$ be a finite subset of $\mathbb{R}^n$, and let $S$ be a nonempty proper subset of $T$. Then $V_T(S) = \emptyset$ iff*

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{cone}\left\{ \begin{pmatrix} t-s \\ \|t\|^2 - \|s\|^2 \end{pmatrix}, \; s \in S, t \in T \setminus S \right\}.$$

**Corollary 3.** *For a finite subset $T$ of $\mathbb{R}^n$ we have*

$$V_T(s) \neq \varnothing \quad \forall s \in T.$$

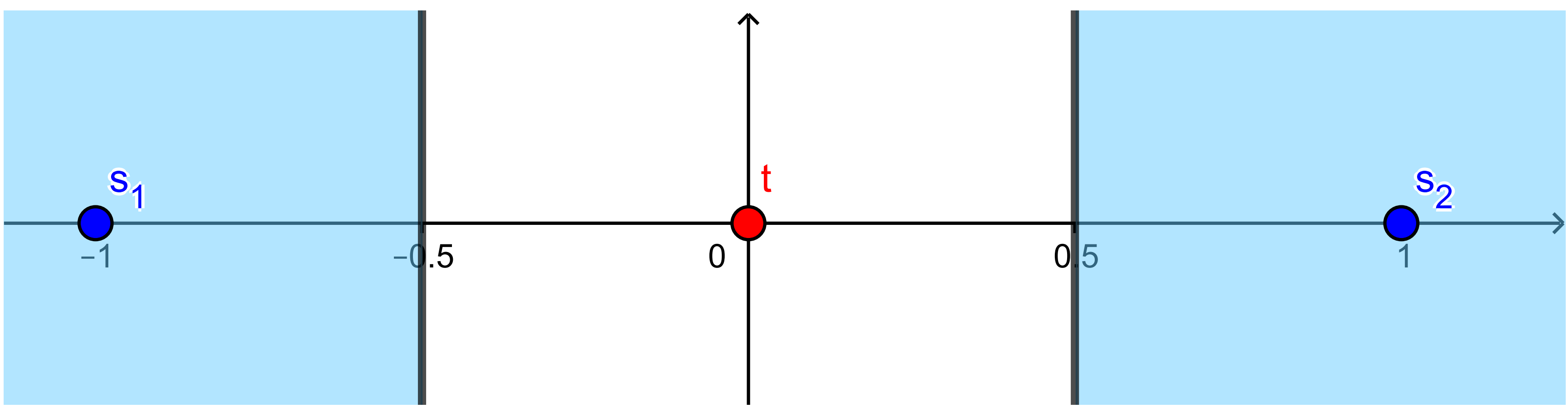
*Moreover $V_T(T \setminus \{t\}) \neq \varnothing$ iff $t$ is an extreme point (vertex) of* conv $T$.

**Corollary 4.** *Let $T$ be a finite subset of $\mathbb{R}^n$, and let $S$ be a nonempty proper subset of $T$. If* $(\operatorname{conv} S) \cap (T \setminus S) \neq \emptyset$, *then $V_T(S) = \emptyset$.*

# Empty cells

**Example 5.** Let $s_1 = (-1, 0),\ s_2 = (1, 0),\ t = (0, 0).$

$$V_T(S) = \left\{(x_1, x_2)\ :\ x_1 \le -\frac{1}{2}, x_1 \ge \frac{1}{2}\right\} = \emptyset.$$



# Bounded cells

**Theorem 6.** *Let $T$ be a finite subset of $\mathbb{R}^n$, and let $S$ be a nonempty proper subset of $T$. Then $V_T(S)$ is bounded iff*

$$\text{cone}\,\{t-s, s \in S, t \in T \setminus S\} = \mathbb{R}^n.$$

It follows from Theorem 6 that if $n \geq 2$ and $|T| \leq 3$ all nonempty cells are unbounded.

**Theorem 7.** *Let $T$ be a finite subset of $\mathbb{R}^n$. If*

$$|T| < 2\sqrt{n+1}, \tag{1}$$

*then for any $S \subset T$ the cell $V_T(S)$ is unbounded.*

**Proposition 8.** *Let $S \subset T \subset \mathbb{R}^n$, with $|S| = 3$ and $|T| = 4$. Then $V_T(S)$ is either empty or unbounded.*

**Proposition 9.** *Let $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$. Then*

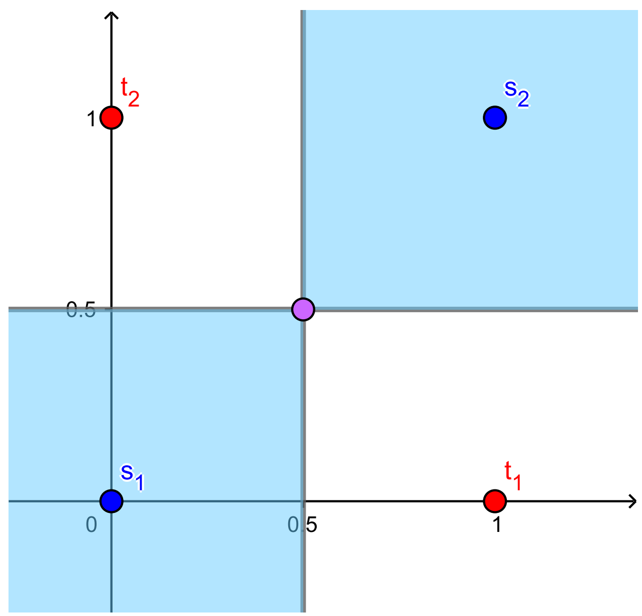
$$V_T(S) \text{ is bounded iff } (s_1, s_2) \cap (t_1, t_2) \neq \emptyset.$$

# Nonempty interior

**Theorem 10.** *Let $T$ be a finite subset of $\mathbb{R}^n$, and let $S$ be a nonempty proper subset of $T$. Then* $\operatorname{int} V_T(S) \neq \varnothing$ *iff*

$$0_{n+1} \notin \operatorname{conv}\left\{ \begin{pmatrix} t - s \\ \|t\|^2 - \|s\|^2 \end{pmatrix}, \quad s \in S,\ t \in T \setminus S \right\}.$$

**Example 11.** $T = \{(0,0),(1,1),(1,0),(0,1)\}, \quad S = \{(0,0),(1,1)\}.$



# Decompositions

**Proposition 12.** *Let $T$ be a finite subset of $\mathbb{R}^n$, and let $S$ be a nonempty proper subset of $T$. If $|S| \geq 2$, then*

$$V_T(S) = \bigcup_{s \in S} \left[ V_{T \setminus \{s\}}(S \setminus \{s\}) \cap V_S(s) \right]. \tag{2}$$

**Proposition 13.** *Let $T$ be a finite subset of $\mathbb{R}^n$, and let $S$ be a nonempty proper subset of $T$. If there exist $s_1, s_2 \in S$ and $t_1, t_2 \in T \setminus S$ such that the inequalities*

$$\|s_1 - x\| \leq \|t_1 - x\| \quad \textit{and} \quad \|s_2 - x\| \leq \|t_2 - x\| \tag{3}$$

*define the same halfspace, then these inequalities are nonessential for $V_T(S)$, i.e. they can be dropped from the system (1).*

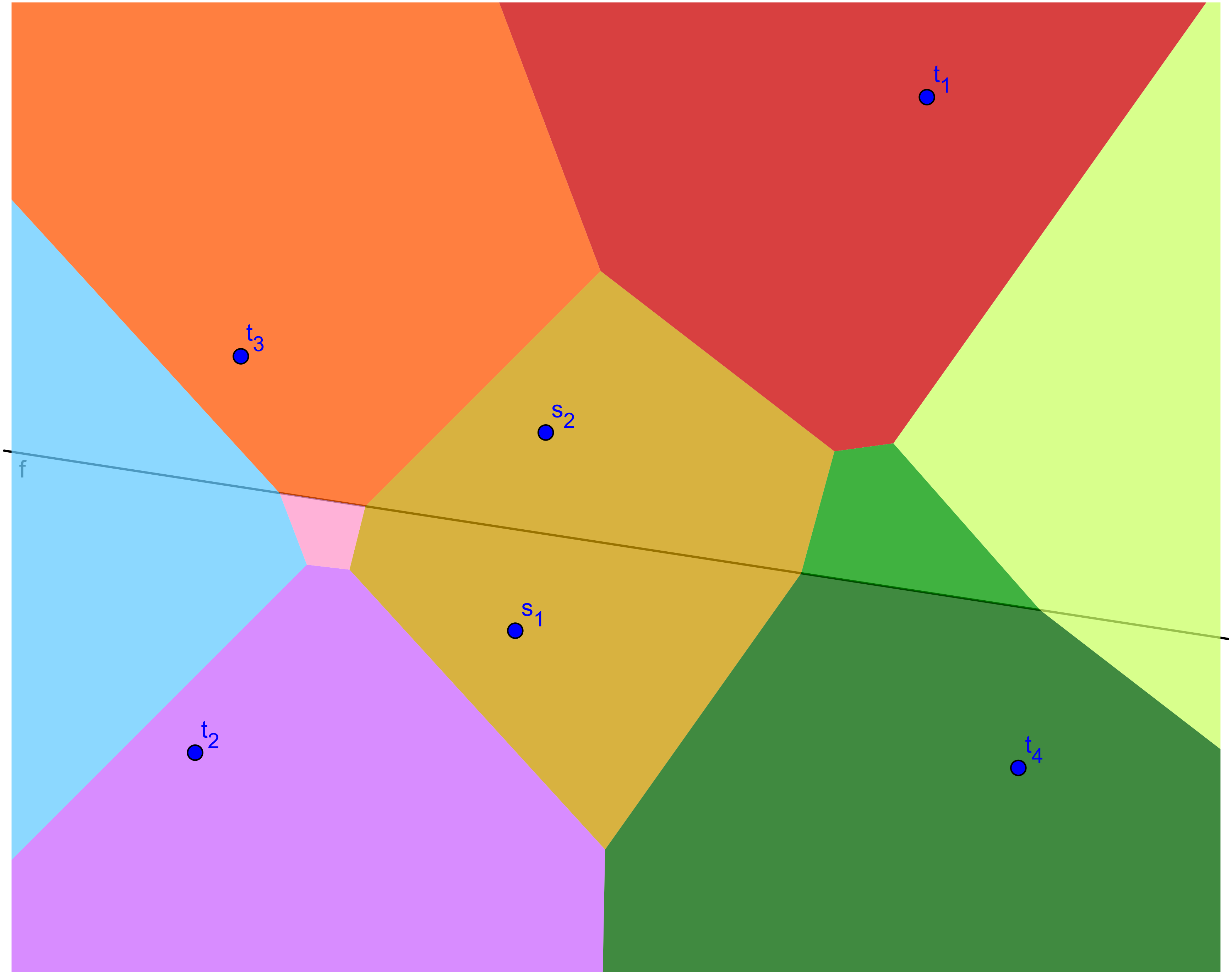
# Decompositions

**Theorem 14.** *Let* $T \subset \mathbb{R}^n$ *be a finite set, let* $S := \{s_1, s_2\} \subset T$ *be a two-point set, and let*

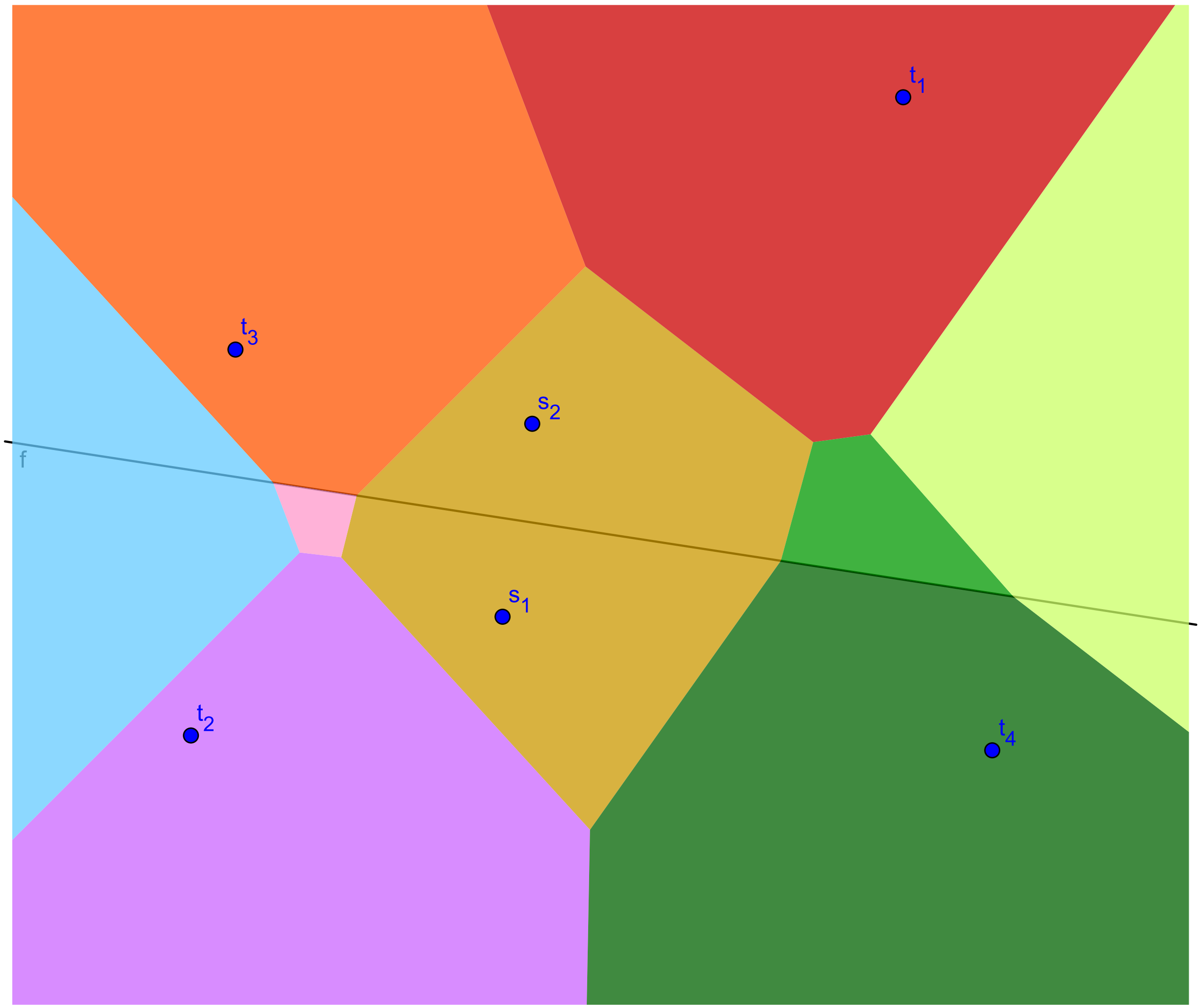
$$H := \{x \in \mathbb{R}^n : \|x - s_1\| = \|x - s_2\|\}$$
$$= \{x \in \mathbb{R}^n : \langle s_1 - s_2, x\rangle = \frac{1}{2}(\|s_1\|^2 - \|s_2\|^2)\}.$$

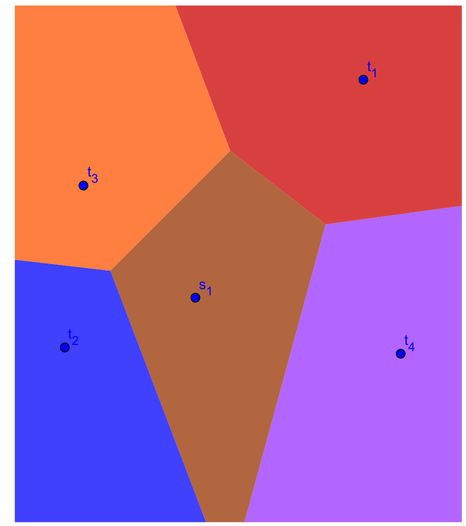
*If* $\operatorname{int} V_T(S) \neq \emptyset$*, then* $H \cap \operatorname{ri} F = \emptyset$ *for every facet* $F$ *of* $V_T(S)$*.*

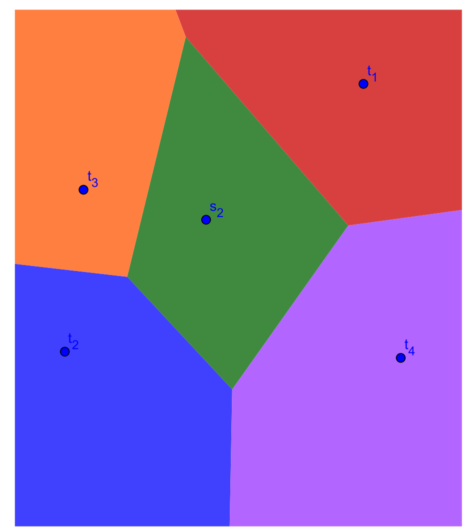
# An example



# An example



# Case study in the plane: $2 \leq |S| < |T| \leq 4$

**Proposition 15.** *Let $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$. The following statements are equivalent:*
*a) $V_T(S)$ is nonempty and at most one-dimensional.*
*b) The points of $T$ are the vertices of a cyclic quadrilateral, with the sites of $S$ located opposite to each other (across a diagonal).*
*c) $V_T(S)$ is a singleton.*

Note that for the case $|S| = 2$ and $|T| = 3$ it is impossible to have a nonempty bounded cell due to Theorem 6: the conic hull of two vectors is always a proper subset of $\mathbb{R}^2$. This means that we do not need to consider this configuration when discussing the subsequent cases of bounded polygons.

Furthermore, in the case $|S| = 3$ and $|T| = 4$ it is impossible to have a bounded cell, as was shown in Proposition 8.

# One-dimensional cells

Since we have determined that we can not have a nonempty bounded cell for $|T| = |S| + 1$, the only possibility to have a singleton cell is for $|S| = 2$ and $|T| = 4$. Furthermore, we can focus on the latter case when studying other bounded cells.

It follows from the preceding discussion that it is impossible to obtain line segments as multipoint Voronoi cells in our setting.

**Corollary 16.** *Let $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$. Then $V_T(S)$ is not one-dimensional.*

It follows from Corollary 16 that it is impossible to have a one-dimensional cell for $|T| = 4$, $|S| = 2$, so both rays and lines are impossible in this configuration.

**Proposition 17.** *Let $S \subset T \subset \mathbb{R}^n$, with $|T| = |S| + 1$. Then $V_T(S)$ is not one-dimensional.*

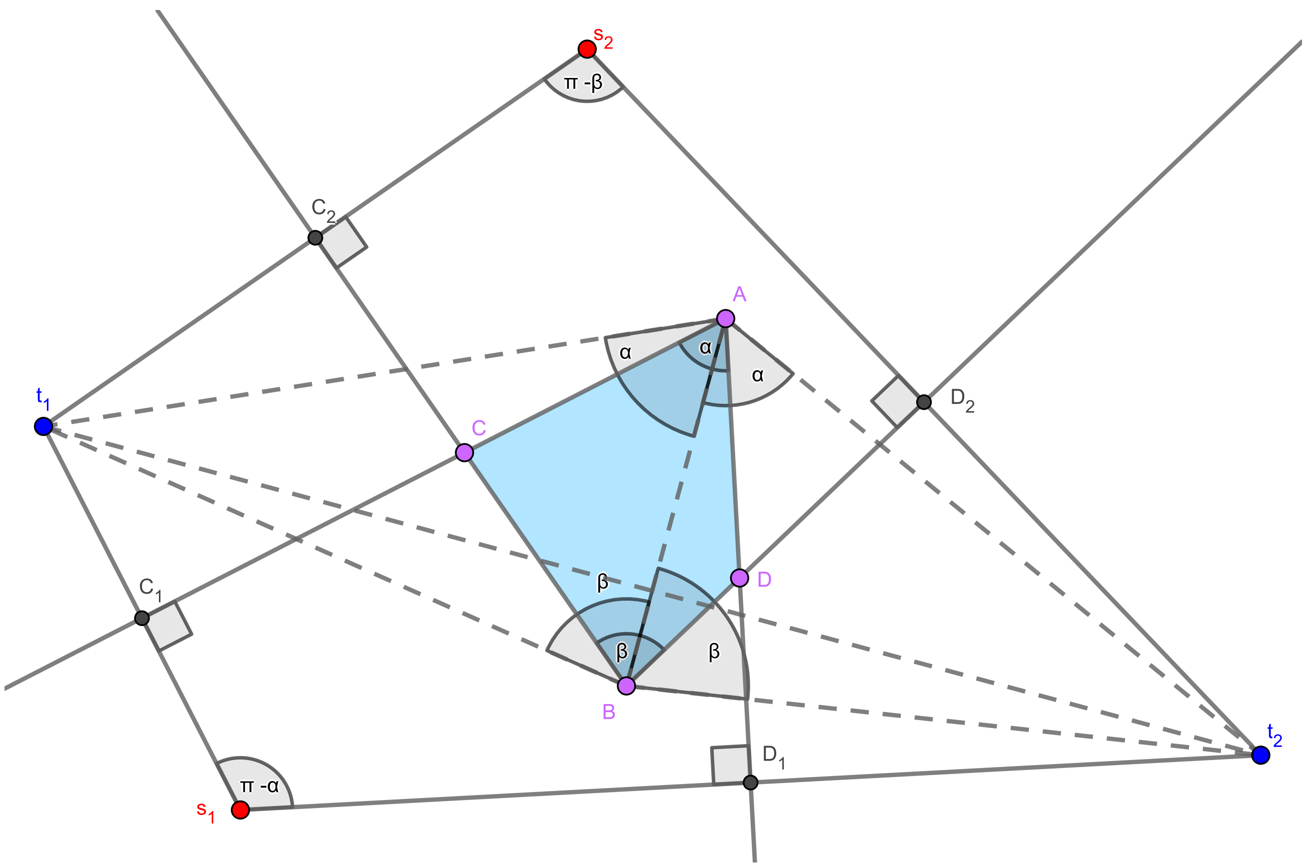
# Triangles

A two-point cell cannot be a triangle. We only need to prove this for the case $|S| = 2$ and $|T| = 4$.

**Proposition 18.** *Let* $S \subset T \subset \mathbb{R}^2$, *with* $|S| = 2$ *and* $|T| = 4$. *Then* $V_T(S)$ *is not a triangle.*

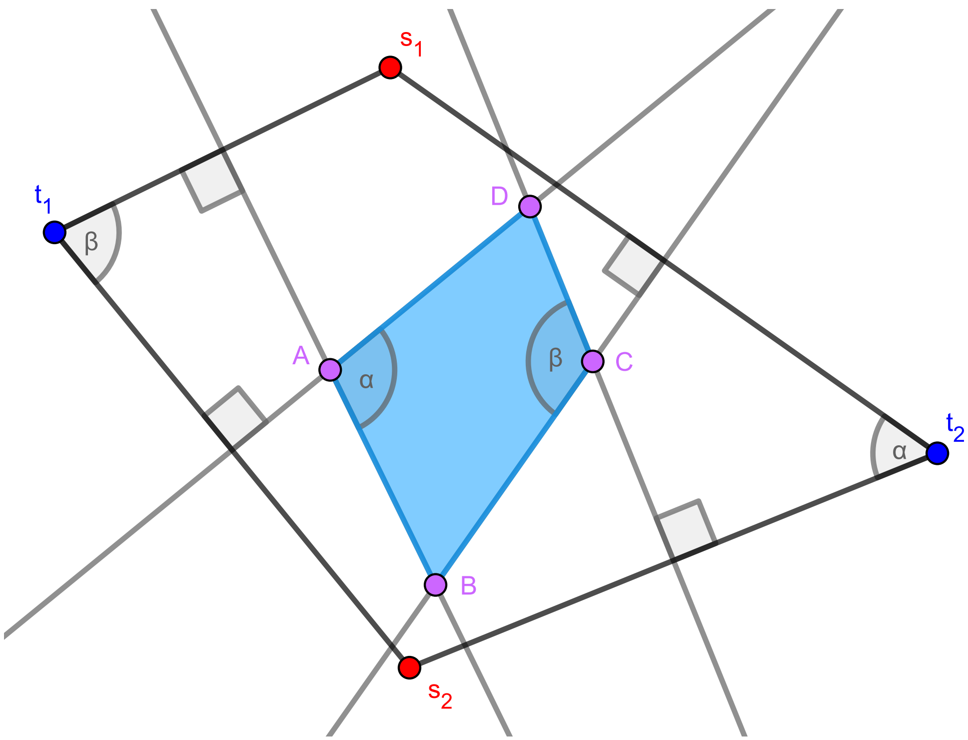
# Bounded quadrilaterals

**Proposition 19.** *Let $F \subset \mathbb{R}^2$ be a non-cyclic bounded quadrilateral. Then there exist $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$, such that $V_T(S) = F$.*

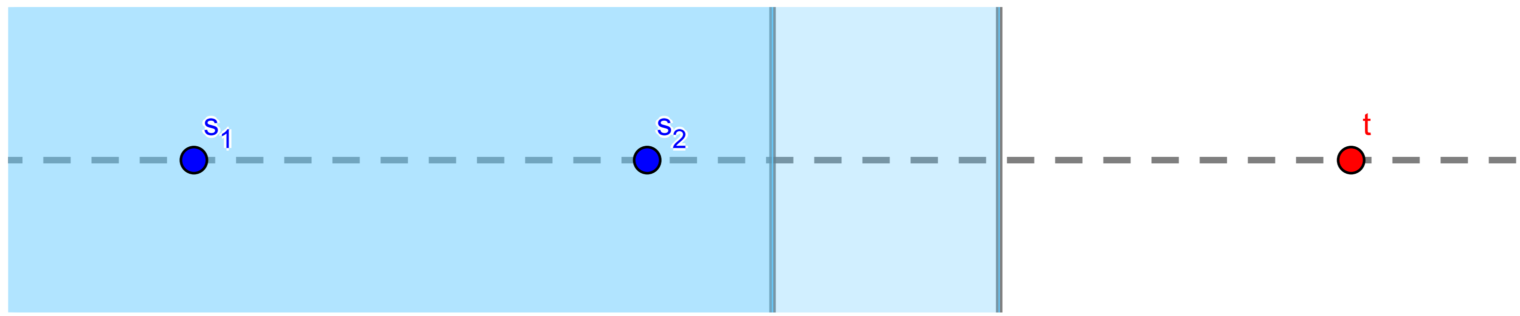


# Bounded quadrilaterals

**Proposition 20.** *Let $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$. Then $V_T(S)$ is not a cyclic quadrilateral.*



# Halfspaces




**Proposition 21.** *Let $F \subset \mathbb{R}^n$ be a halfspace. Then for any two integers $\tau > \sigma \geq 1$ there exist $S \subset T \subset \mathbb{R}^n$, with $|S| = \sigma$ and $|T| = \tau$, such that $V_T(S) = F$.*

# Intersections of parallel halfspaces

**Proposition 22.** *Let* $F \subset \mathbb{R}^n$ *be a nonempty intersection of two parallel halfspaces with opposite normals. Then there exist* $S \subset T \subset \mathbb{R}^n$*, with* $|S| = 2$ *and* $|T| = 4$*, such that* $V_T(S) = F$*.*

# Wedges

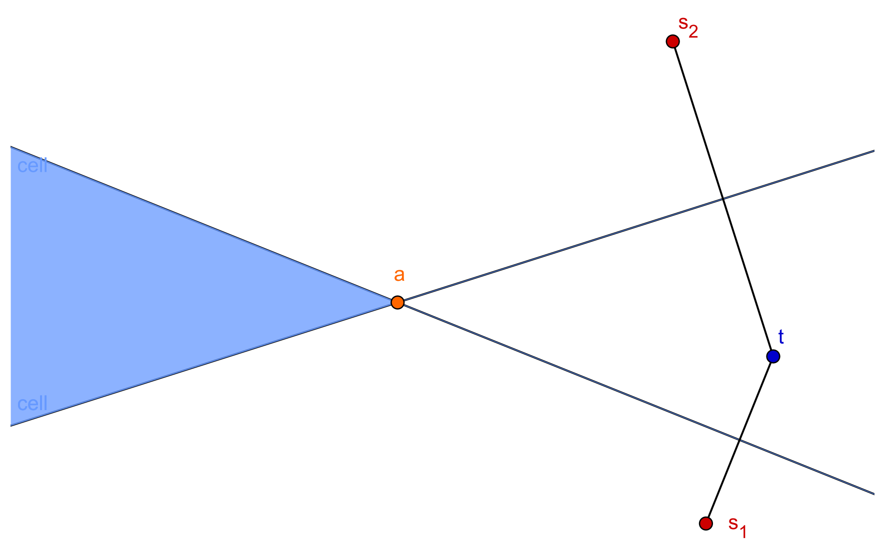
If $V$ is the intersection of two non-parallel halfplanes,

$$V = \{x \in \mathbb{R}^2 \,|\, \langle x, v_1 \rangle \leq \langle a, v_1 \rangle,\ \langle x, v_1 \rangle \leq \langle a, v_2 \rangle\},$$

where $v_1, v_2, a \in \mathbb{R}^2$, and $\{v_1, v_2\}$ is a linearly independent system, we can choose any point $t$ in the set

$$V' = \{x \in \mathbb{R}^2 \,|\, \langle x, v_1 \rangle > \langle a, v_1 \rangle,\ \langle x, v_1 \rangle > \langle a, v_2 \rangle\},$$

and let $s_1$ and $s_2$ be the two reflections of $t$ with respect to the lines $\langle x, v_i \rangle = \langle a, v_i \rangle$, $i = 1, 2$.



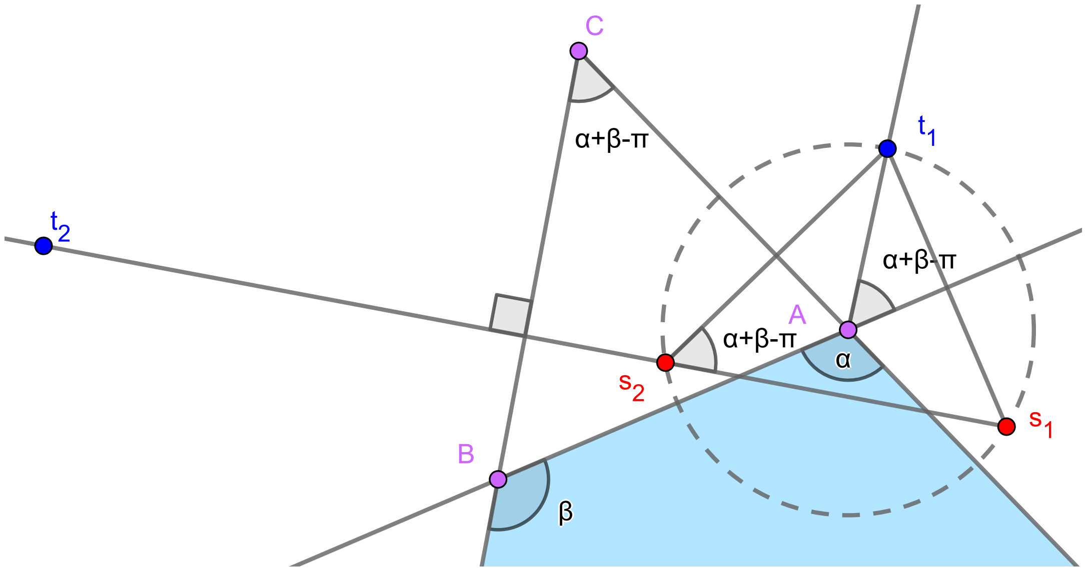
# Wedges

**Proposition 23.** *Let*

$$F := \{x \in \mathbb{R}^n \,|\, \langle x, v_i \rangle \le b_i \qquad (i = 1, 2)\}\,,$$

*with* $v_1, v_2 \in \mathbb{R}^n$ *linearly independent unit vectors and* $b_1, b_2 \in \mathbb{R}$. *Then there exist* $S \subset T \subset \mathbb{R}^n$, *with* $|S| = 2$ *and* $|T| = 4$, *such that* $V_T(S) = F$.

# Unbounded polygons with three sides



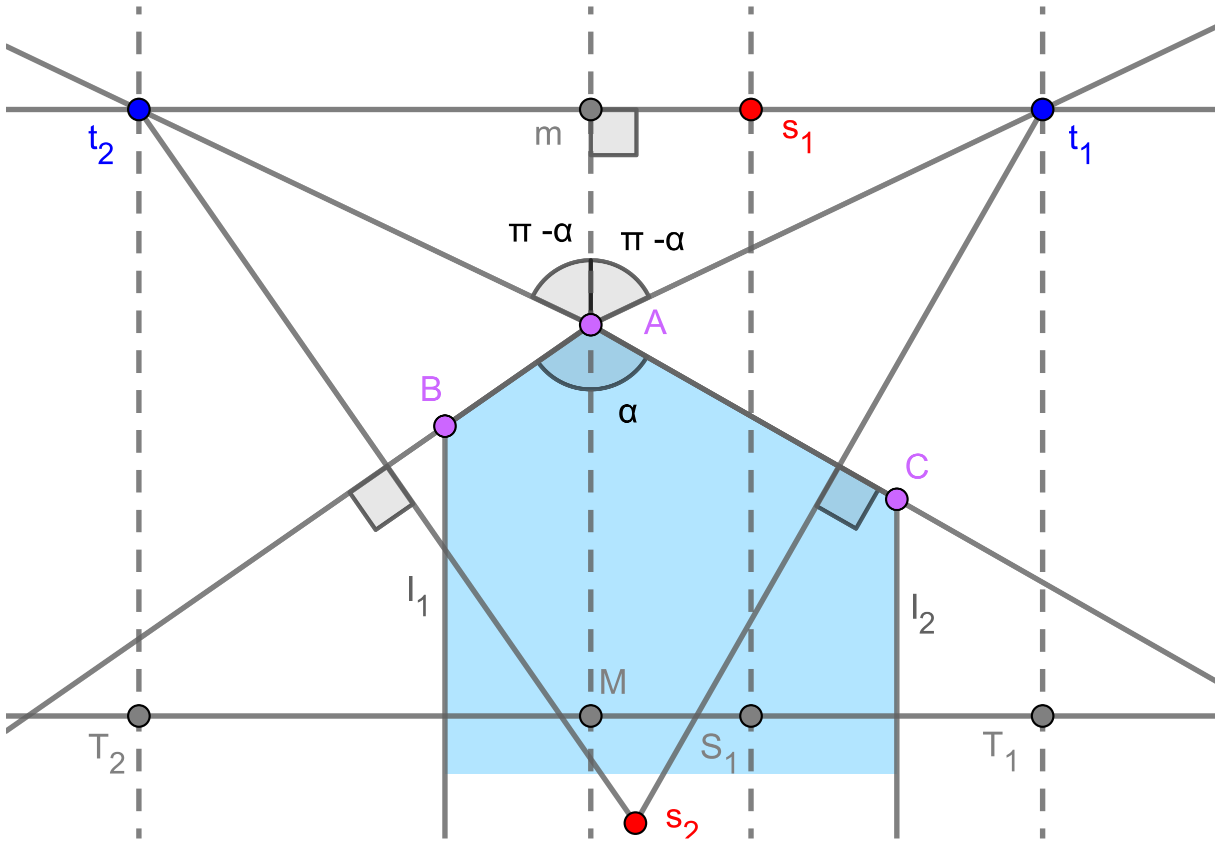

**Proposition 24.** *Let $F$ be an unbounded polygon with three (non-parallel) sides. Then there exist $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$, such that $V_T(S) = F$.*

**Proposition 25.** *Let $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$. Then $V_T(S)$ is not an unbounded polygon with parallel sides and just two vertices.*

# Unbounded polygons with three sides

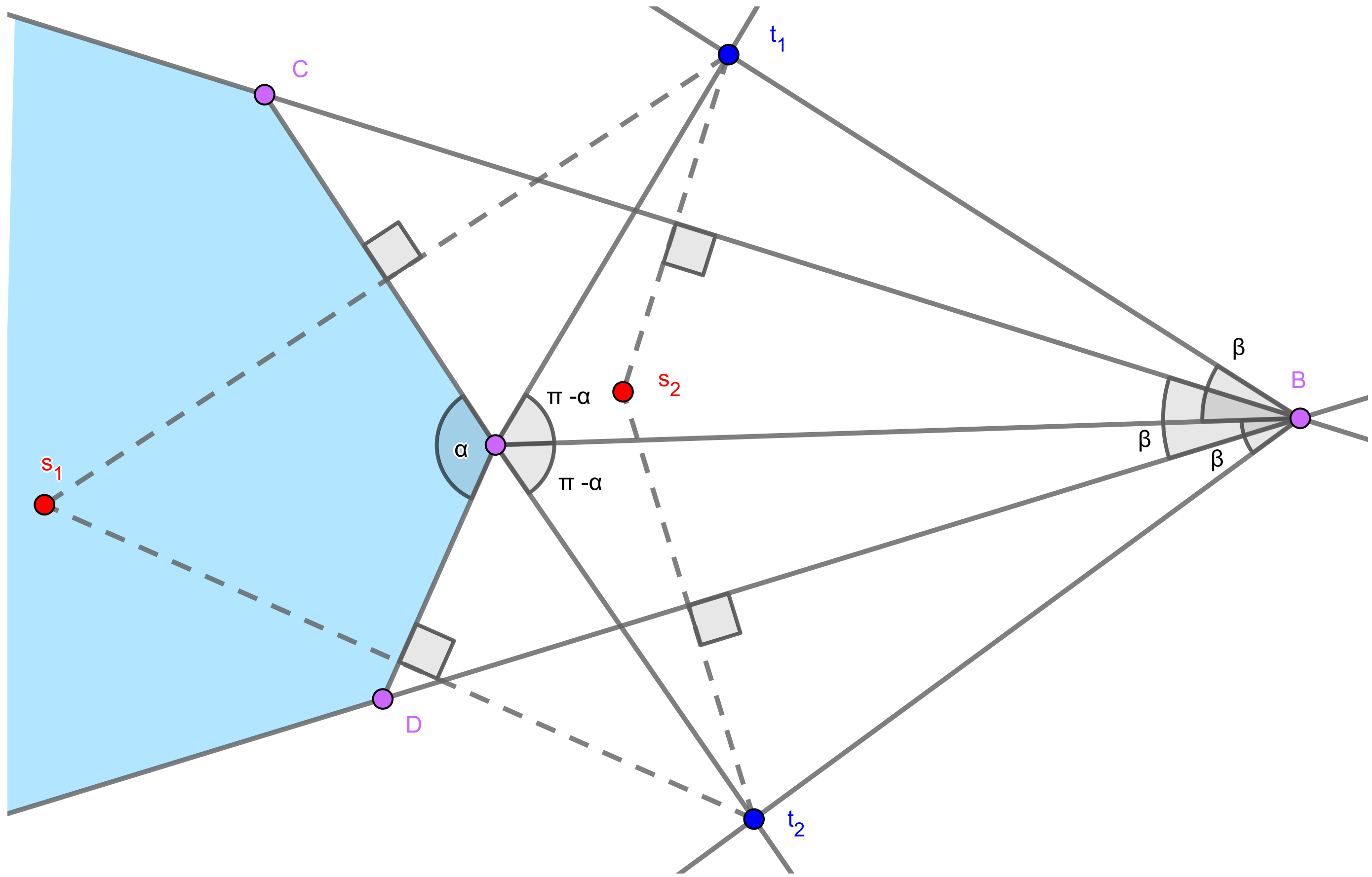
We note here that it is possible to have an unbounded polygon with three sides for the case $|S| = 3$ and $|T| = 4$ if and only if the unbounded sides are non-parallel. Indeed, in this case we can first build a wedge that defines the two unbounded sides, and then add an extra site to define the extra inequality. For the case of unbounded parallel sides, it is clear that the point $t$ should at the same time lie outside of each of these parallel sides, which is impossible.

# Unbounded quadrilateral #1




**Proposition 26.** *Let $F$ be an unbounded quadrilateral with two parallel sides. Then there exist $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$, such that $V_T(S) = F$.*

# Unbounded quadrilateral #2




**Proposition 27.** *Let $F$ be an unbounded quadrilateral with no sides parallel. Then there exist $S \subset T \subset \mathbb{R}^2$, with $|S| = 2$ and $|T| = 4$, such that $V_T(S) = F$.*

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Thank you