Necessary and sufficient conditions for globally best Chebyshev approximation

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The problem

Consider a family of functions $\Gamma \subset C(\mathbb{R})$. We want to approximate a continuous function f by a function $g \in \Gamma$ over an interval $[a, b]$. We use uniform norm $\|f - g\|_\infty = \sup_{t \in [a,b]} |f(t) - g(t)|$.

Today we are interested in approximating f using a continuous piecewise polynomial.

Other families of functions

For some families of functions (polynomials, trigonometric polynomials), the problem is largely solved. Approaches include:

- Algebraic
- Analytic
- Geometric

All approaches rely on the fundamental theorem of algebra, and characterisations are given in terms of alternating sequences.

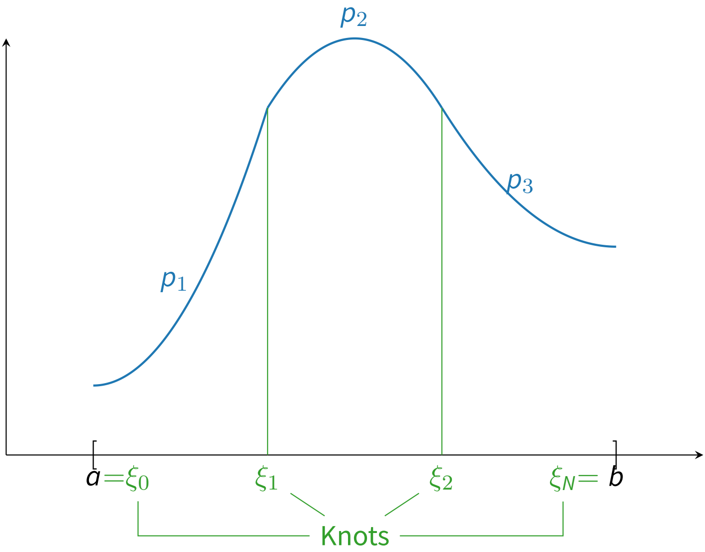
Piecewise polynomials

s is a piecewise polynomial (spline) over the interval $[a, b]$, then there exists *knots* $a = \xi_0 \leq \xi_1 \leq \ldots \leq \xi_m \leq \xi_{m+1} = b$ and polynomials $p_0, \ldots, p_m$ such that $s(t) = p_i(t)$ for any $t \in [\xi_i, \xi_{i+1}]$ and any $i = 0, \ldots, m$. The degree n of s is the maximum degree amongst the polynomials $p_0, \ldots, p_m$.

We assume that a bound on the degree n and the number of pieces $m + 1$ are known, but not the location of the knots.

The fundamental theorem of algebra doesn't apply.

Background



Notations and formulation

Denote by $T = [a, b]^m$ the set of possible locations of m knots in $[a, b]$ and Π_n the set of polynomials of degree at most n.

$$\Gamma = \{s \in C([a, b]) : \exists(\xi_1 \leq \ldots \leq \xi_m) \in T, \\ p_0, \ldots, p_m \in \Pi_n, s \restriction [\xi_i, \xi_{i+1}] = p_i, i = 0, \ldots, m\}$$

$$\text{minimise } \|f - s\|_\infty \text{ subject to } s \in \Gamma$$

Existing results

- Existence (Schumaker 1968)
- Local optimality conditions (Nürnberger, Schumaker, Sommer and Strauss 1989; Sukhorukova and JU 2017)
- Sufficent global conditions (Nürnberger 1989)

Fixed knots formulation

First suppose that the knots $\xi_1, \ldots, x_m$ are known. T

$$\text{minimise } u \text{ subject to}$$

$$\begin{aligned}
u - \sum_{i=0}^{n} a_{i,j} t^i &\geq f(t) && \forall t \in [\xi_i, \xi_{i+1}], j = 0, \ldots, m \\
u + \sum_{i=0}^{n} a_{i,j} t^i &\geq -f(t) && \forall t \in [\xi_i, \xi_{i+1}], j = 0, \ldots, m \\
\sum_{i=0}^{n} a_{i,j} \xi_j^i - \sum_{i=0}^{n} a_{i,j-1} \xi_j^i &= 0 && j = 0, \ldots, m
\end{aligned} \tag{P_Ξ}$$

Dual Formulation

$$\text{maximise} \sum_{t\in[a,b]} (y_t^+ - y_t^-) f(t) \text{ subject to} \qquad (D_\Xi)$$

$$\sum_{t\in[a,b]} (y_t^+ + y_t^-) = 1$$

$$\sum_{t\in[\xi_j,\xi_{j+1}]} (y_t^+ - y_t^-) t^i - z_j \xi_j^i + z_{j-1} \xi_{j-1}^i = 0 \quad j = 1, \dots, n-1$$

$$\sum_{t\in[\xi_n,\xi_{n+1}]} (y_t^+ - y_t^-) t^i + z_j \xi_n^i = 0 \qquad (C_\Xi)$$

$$\sum_{t\in[\xi_0,\xi_1]} (y_t^+ - y_t^-) t^i - z_j \xi_1^i = 0$$

$$y_t^+ \geq 0, y_t^- \geq 0 \qquad t \in [a,b]$$

and a finite number of y_t are positive.

Premilinary results

Any feasible solution to the dual is given by a nontrivial solution to the system $Wx = 0$, for W of the form: The matrix

$$W = \begin{pmatrix} \boxed{V_1} & 0 & \dots\dots\dots\dots & & 0 \\ 0 & \boxed{V_2} & 0 & \dots\dots\dots & 0 \\ 0 & 0 & \boxed{V_3} & 0 \quad \cdots & 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 0 & \dots\dots\dots & 0 & \boxed{V_n} & \end{pmatrix}$$

The blocks V_i are Vandermonde-like.

Remark

$$\text{minimise} \sum_{t\in[a,b]} (y_t^+ - y_t^-) f(t) \text{ subject to}$$

$$\sum_{t\in[a,b]} (y_t^+ + y_t^-) = 1$$

$$\sum_{t\in[\xi_j,\xi_{j+1}]} (y_t^+ - y_t^-) t^i - z_j \xi_j^i + z_{j-1} \xi_{j-1}^i = 0 \quad j = 1, \ldots, n-1$$

$$\sum_{t\in[\xi_n,\xi_{n+1}]} (y_t^+ - y_t^-) t^i + z_j \xi_n^i = 0$$

$$\sum_{t\in[\xi_0,\xi_1]} (y_t^+ - y_t^-) t^i - z_j \xi_1^i = 0$$

$$y_t^+ \geq 0, y_t^- \geq 0 \qquad t \in [a, b]$$

and a finite number of y_t are positive.

Remark

$$\text{maximise } -\sum_{t\in[a,b]} (y_t^+ - y_t^-)f(t) \text{ subject to}$$

$$\sum_{t\in[a,b]} (y_t^+ + y_t^-) = 1$$

$$\sum_{t\in[\xi_j,\xi_{j+1}]} (y_t^+ - y_t^-)t^i - z_j\xi_j^i + z_{j-1}\xi_{j-1}^i = 0 \quad j = 1,\ldots,n-1$$

$$\sum_{t\in[\xi_n,\xi_{n+1}]} (y_t^+ - y_t^-)t^i + z_j\xi_n^i = 0$$

$$\sum_{t\in[\xi_0,\xi_1]} (y_t^+ - y_t^-)t^i - z_j\xi_1^i = 0$$

$$y_t^+ \geq 0, y_t^- \geq 0 \qquad t \in [a,b]$$

and a finite number of y_t are positive.

Remark

$$\text{maximise} \sum_{t\in[a,b]} (y_t^+ - y_t^-)(-f(t)) \text{ subject to}$$

$$\sum_{t\in[a,b]} (y_t^+ + y_t^-) = 1$$

$$\sum_{t\in[\xi_j,\xi_{j+1}]} (y_t^+ - y_t^-)t^i - z_j\xi_j^i + z_{j-1}\xi_{j-1}^i = 0 \quad j = 1,\ldots,n-1$$

$$\sum_{t\in[\xi_n,\xi_{n+1}]} (y_t^+ - y_t^-)t^i + z_j\xi_n^i = 0$$

$$\sum_{t\in[\xi_0,\xi_1]} (y_t^+ - y_t^-)t^i - z_j\xi_1^i = 0$$

$$y_t^+ \geq 0, y_t^- \geq 0 \qquad t \in [a,b]$$

and a finite number of y_t are positive.

Remark

- the dual problem (D_Ξ)-(C_Ξ) is symmetric: if the maximum is u^*_Ξ, then the minimum is $-u^*_\Xi$.
- There is a feasible solution taking any value $[-u^*_\Xi, u^*_\Xi]$,

Comparing with the solution

Let

$$u^* = \min_{s \in \Gamma} \|f - s\|_\infty$$
$$u^* = \|f - s^*\|_\infty$$

$\Xi^* = (\xi_1^*, \ldots, \xi_m^*)$ the knots of s^* and (y^*, z^*) the optimal dual variable of (D_Ξ).

Then $u^* \in [-u_\Xi^*, u_\Xi^*]$.

An upper bound

$$u^* \in [-u^*_\Xi, u^*_\Xi] \implies \exists u_\Xi \in [-u^*_\Xi, u^*_\Xi], u^* \leq u_\Xi \forall \Xi \in T$$

We can estimate u^* from above by solving the problem:

$$\text{maximise } u \text{ subject to } u \in [-u^*_\Xi, u^*_\Xi]$$

A upper bound

We can estimate u^* from above by solving the problem:

$$\text{maximise } u \text{ subject to} \tag{D}$$

$$\begin{aligned}
\sum_{t\in[a,b]} (y^+_{t,\Xi} - y^i_{t,\Xi}) f(t) - u &= 0 && \Xi \in T \\
\sum_{t\in[a,b]} (y^+_{t,\Xi} + y^-_{t,\Xi}) &= 1 && \Xi \in T \\
\sum_{t\in[\xi_j,\xi_{j+1}]} (y^+_{t,\Xi} - y^-_{t,\Xi}) t^i - z_{j,\Xi}\xi^i_j + z_{,\Xi j-1}\xi^i_{j-1} &= 0 && j = 1,\dots,n-1, \Xi \in T \\
\sum_{t\in[\xi_n,\xi_{n+1}]} (y^+_{t,\Xi} - y^-_{t,\Xi}) t^i + z_{j,\Xi}\xi^i_n &= 0 && \Xi \in T \\
\sum_{t\in[\xi_0,\xi_1]} (y^+_{t,\Xi} - y^-_{t,\Xi}) t^i - z_{j,\Xi}\xi^i_1 &= 0 && \Xi \in T \\
y^+_{t,\Xi} \geq 0, y^-_{t,\Xi} &\geq 0 && t \in [a,b], \Xi \in T
\end{aligned} \tag{C}$$

and a finite number of $y_{t,\Xi}$ are positive for each Ξ. (1)

A upper bound

We can estimate u^* from above by solving the problem:

$$\text{maximise } u \text{ subject to} \qquad (D)$$

$$\sum_{t\in[a,b]} (y^+_{t,\Xi} - y^i_{t,\Xi}) f(t) - u = 0 \qquad \Xi \in T$$

$$\sum_{t\in[a,b]} (y^+_{t,\Xi} + y^-_{t,\Xi}) = 1 \qquad \Xi \in T$$

$$\sum_{t\in[\xi_j,\xi_{j+1}]} (y^+_{t,\Xi} - y^-_{t,\Xi}) t^i - z_{j,\Xi}\xi^i_j + z_{,\Xi j-1}\xi^i_{j-1} = 0 \qquad j = 1,\dots,n-1, \Xi \in T \qquad (C)$$

$$\sum_{t\in[\xi_n,\xi_{n+1}]} (y^+_{t,\Xi} - y^-_{t,\Xi}) t^i + z_{j,\Xi}\xi^i_n = 0 \qquad \Xi \in T$$

$$\sum_{t\in[\xi_0,\xi_1]} (y^+_{t,\Xi} - y^-_{t,\Xi}) t^i - z_{j,\Xi}\xi^i_1 = 0 \qquad \Xi \in T$$

$$y^+_{t,\Xi} \geq 0, y^-_{t,\Xi} \geq 0 \qquad t \in [a,b], \Xi \in T$$

and a finite number of $y_{t,\Xi}$ are positive for each Ξ. (1)

A upper bound

We can estimate u^* from above by solving the problem:

$$\text{maximise } u \text{ subject to} \tag{D}$$

$$\sum_{t\in[a,b]} (y^+_{t,\Xi} - y^i_{t,\Xi})f(t) - u = 0 \qquad \Xi \in T$$

$$\sum_{t\in[a,b]} (y^+_{t,\Xi} + y^-_{t,\Xi}) = 1 \qquad \Xi \in T$$

$$\sum_{t\in[\xi_j,\xi_{j+1}]} (y^+_{t,\Xi} - y^-_{t,\Xi})t^i - z_{j,\Xi}\xi^i_j + z_{,\Xi j-1}\xi^i_{j-1} = 0 \qquad j = 1,\dots,n-1, \Xi \in T \tag{C}$$

$$\sum_{t\in[\xi_n,\xi_{n+1}]} (y^+_{t,\Xi} - y^-_{t,\Xi})t^i + z_{j,\Xi}\xi^i_n = 0 \qquad \Xi \in T$$

$$\sum_{t\in[\xi_0,\xi_1]} (y^+_{t,\Xi} - y^-_{t,\Xi})t^i - z_{j,\Xi}\xi^i_1 = 0 \qquad \Xi \in T$$

$$y^+_{t,\Xi} \geq 0, y^-_{t,\Xi} \geq 0 \qquad t \in [a,b], \Xi \in T$$

and a finite number of $y_{t,\Xi}$ are positive for each Ξ. (1)

Dual IDLP

The dual problem of (D) is

$$\text{minimise } u \text{ subject to}$$

$$\begin{aligned} u - \sum_{i=0}^{n} a_{i,j,\Xi} t^i - w_\Xi f(t) &\geq 0 \quad \forall t \in [\xi_i, \xi_{i+1}], i = 0, \ldots, m, \Xi \in T \\ u + \sum_{i=0}^{n} a_{i,j,\Xi} t^i + w_\Xi f(t) &\geq 0 \quad \forall t \in [\xi_i, \xi_{i+1}], i = 0, \ldots, m \\ \sum_{i=0}^{n} a_{i,j,\Xi} \xi_j^i - \sum_{i=0}^{n} a_{i,j,\Xi-1} \xi_j^i &= 0 \quad i = 0, \ldots, m \\ \sum_{\Xi \in T} w_\Xi &= 1 \end{aligned} \qquad (P)$$

IDLP

- Weak duality holds: if u_D and $u_{\P}$ are feasible solutions to (D) and (P), then $u_D \leq u_P$.
- A duality gap may exist

Binary formulation

u^* is the solution to the following problem:

$$\text{minimise } u \text{ subject to}$$

$$\begin{aligned}
u - \sum_{i=0}^{n} a_{i,j,\Xi} t^i - w_\Xi f(t) &\geq 0 && \forall t \in [\xi_i, \xi_{i+1}], i = 0, \ldots, m, \Xi \in T \\
u + \sum_{i=0}^{n} a_{i,j,\Xi} t^i + w_\Xi f(t) &\geq 0 && \forall t \in [\xi_i, \xi_{i+1}], i = 0, \ldots, m \\
\sum_{i=0}^{n} a_{i,j,\Xi} \xi_j^i - \sum_{i=0}^{n} a_{i,j,\Xi-1} \xi_j^i &= 0 && i = 0, \ldots, m \\
\sum_{\Xi \in T} w_\Xi &= 1 \\
w_\Xi &\in \{0, 1\} && \Xi \in T
\end{aligned} \tag{IP}$$

No Duality Gap

Proposition

Let u_D^*, u_P^* *and* u_I^* *be the respective solutions of* (D), (P) *and* (IP). *Then*

$$u_I^* = u^* \leq u_D^* \leq u_P^* \leq u_I^*$$

Bisection algorithm

1. Select a set of knots $\Xi \in T$ and find the solution u_Ξ to the problem (P_Ξ). Let $u^+ = u_\Xi$ and $u^- = 0$.
2. Set
$$u_k = \frac{u_k^+ + u_k^-}{2}$$
and solve the system (C).
3. - If the system is feasible, then this provides a feasible solution to the Problem (D_Ξ)-(C_Ξ), and u_k is a lower bound to the best approximation. Set $u_{k+1}^+ = u_k^+$ and $u_{k+1}^- = u_k$. Set $k = k+1$ and go to Step 2.
   - Otherwise there is no feasible solution for the value u_k, and it provides an upper bound to the best approximation. Set $u_{k+1}^+ = u_k$ and $u_{k+1}^- = u_k^-$. Set $k = k+1$ and go to Step 2.

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