Nonlinear transversality of collections of sets

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Outline

1. Nonlinear transversality properties

2. Metric characterizations

3. Dual characterizations

4. Sets and set-valued mappings

Transversality

$X-$normed space, $\Omega_1, \Omega_2 \subset X$, $\bar{x} \in \Omega_1 \cap \Omega_2$.

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- Transversality properties $\longleftrightarrow$ 'good' arrangements.
- Applications:
  - calculus rules for normal cones, subdifferentials, coderivatives, optimality conditions.
  - convergence analysis of computational algorithms.

$\varphi-$semitransversality

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(0) = 0,\ \varphi'_+(0) \geq 0, \text{ and } \varphi'(t) > 0 \text{ for all } t > 0.$$

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Definition

(i) $\varphi-$semitransversal:

$$(\Omega_1 - x_1) \cap (\Omega_2 - x_2) \cap B_\rho(\bar{x}) \neq \emptyset$$

for all $\rho \in]0, \delta[$ and $x_1, x_2 \in \varphi^{-1}(\rho)\mathbb{B}$.

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Definition

(ii) $\varphi-$subtransversal:

$$(\Omega_1 + \varphi^{-1}(\rho)\mathbb{B}) \cap (\Omega_2 + \varphi^{-1}(\rho)\mathbb{B}) \cap B_\delta(\bar{x}) \subset (\Omega_1 \cap \Omega_2) + \rho\mathbb{B}$$

for all $\rho \in]0, \delta[$.

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(iii) $\varphi-$transversal:

$$(\Omega_1 - \omega_1 - x_1) \cap (\Omega_2 - \omega_2 - x_2) \cap (\rho\mathbb{B}) \neq \emptyset,$$

for all $\rho \in]0, \delta[$, $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$, $x_i \in \varphi^{-1}(\rho)\mathbb{B}$ $(i = 1, 2)$.

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$\varphi-$semitransversality $\longleftarrow$ $\varphi-$transversality $\longrightarrow$ $\varphi-$subtransversality

Metric characterizations: $\varphi-$semitransversality

Theorem

(i) $\{\Omega_1, \Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$ if and only if $\exists \delta > 0$ such that

$$d(\bar{x}, (\Omega_1 - x_1) \cap (\Omega_2 - x_2)) \leq \varphi(\max\{\|x_1\|, \|x_2\|\}), \quad \forall x_1, x_2 \in \delta\mathbb{B}.$$

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+ $\varphi(t) = \alpha t$: semiregularity (Kruger, Thao, 2014).

+ $\varphi(t) = \alpha t^q$ $(0 < q \leq 1)$: $q-$semiregularity (Kruger, Thao, 2015).

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(ii) $\{\Omega_1, \Omega_2\}$ is $\varphi-$subtransversal at $\bar{x}$ if and only if $\exists \delta > 0$ such that

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$+\ \varphi(t) = \alpha t$: (Dolecki, 1982); (Ioffe, 1989); (local) linear regularity (Bauschke, Borwein, 1993); linear estimate, linear coherence (Penot, 1998, 2013); metric inequality (Ngai, Théra, 2001); subregularity (Kruger, Thao, 2014); subtransversality (Ioffe, 2016).

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for all $x \in B_\delta(\bar{x}), x_1, x_2 \in \delta\mathbb{B}$, or equivalently,

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for all $x_1, x_2 \in \delta\mathbb{B}$.

+ $\varphi(t) = \alpha t$: strong metric inequality (Kruger, 2005).

+ $\varphi(t) = \alpha t^q$ $(0 < q \leq 1)$: $q-$regularity (Kruger, Thao, 2015).

Dual characterizations: basic tools

$X-$normed space, $\Omega \subset X$, $f : X \to \mathbb{R} \cup \{+\infty\}$.

- Fréchet normal cone to Ω at $\bar{x} \in \Omega$:

$$N_\Omega(\bar{x}) := \left\{ x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}.$$

- Fréchet subdifferential of f at $\bar{x} \in \operatorname{dom} f$:

$$\partial f(\bar{x}) := \left\{ x^* \in X^* \mid \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

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- Ekeland variational principle (Ekeland, 1974).
- Fuzzy sum rule (Fabian, 1989).

Dual characterizations: $\varphi-$subtransversality

$X-$Asplund, Ω_1, Ω_2 closed, $\bar{x} \in \Omega_1 \cap \Omega_2$.

Theorem (a sufficient condition)

If $\exists \delta > 0$ s.t. $\|x_1^* + x_2^*\| \geq 1$ for all $x \in B_\delta(\bar{x})$, $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$ $(i = 1, 2)$ with $0 < \max_{i=1,2} \|\omega_i - x\| < \delta$ and for all $x' \in B_\delta(x)$, $\omega_i' \in \Omega_i \cap B_\delta(\omega_i)$, $x_i \in B_\delta(\omega_i)$ $(i = 1, 2)$, $(-x_1^*, -x_2^*, x^*) \in \partial\varphi\left(\max_{i=1,2} \|x' - x_i\|\right)$ with

$$0 < \max_{i=1,2} \|x' - x_i\| < \delta, \|x_1^*\| + \|x_2^*\| = \varphi'\left(\max_{i=1,2} \|x' - x_i\|\right) > 0,$$

$$\langle x_i^*, x' - x_i\rangle = \|x_i^*\|\|x' - x_i\|, \; d(x_i^*, N_{\Omega_i}(\omega_i')) < \delta \;\; (i = 1, 2).$$

then $\{\Omega_1, \Omega_2\}$ is $\varphi-$subtransversal at $\bar{x}$.

Sets and set-valued mappings

$X, Y-$metric spaces, $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$.

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Definition

1. metrically $\varphi-$semiregular:

$$d(\bar{x}, F^{-1}(y)) \leq \varphi(d(y, \bar{y})) \;\; \forall y \in B_\delta(\bar{y}).$$

2. metrically $\varphi-$subregular:

$$d(x, F^{-1}(\bar{y})) \leq \varphi(d(\bar{y}, F(x))) \;\; \forall x \in B_\delta(\bar{x}).$$

3. metrically $\varphi-$regular:

$$d(x, F^{-1}(y)) \leq \varphi(d(y, F(x))) \;\; \forall x \in B_\delta(\bar{x}), y \in B_\delta(\bar{y}).$$

Sets and set-valued mappings

$X-$normed space, $\bar{x} \in \Omega_1 \cap \Omega_2$. Consider $F : X \rightrightarrows X^2$ defined by

$$F(x) := (\Omega_1 - x) \times (\Omega_2 - x) \text{ for all } x \in X. \quad \text{(Ioffe, 2000)}$$

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+ $(\bar{x}, (0,0)) \in \mathrm{gph}\, F$.
+ $F^{-1}(x_1, x_2) = (\Omega_1 - x_1) \cap (\Omega_2 - x_2)$ for any $(x_1, x_2) \in X^2$.

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Theorem

$\{\Omega_1, \Omega_2\}$ has a $\varphi-$transversality property at $\bar{x}$ if and only if F has the corresponding $\varphi-$regularity property at $(\bar{x}, (0,0))$.

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$$\Omega_1 := \operatorname{gph} F, \;\; \Omega_2 := X \times \{\bar{y}\} \longrightarrow (\bar{x}, \bar{y}) \in \Omega_1 \cap \Omega_2.$$

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$\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi^{-1}(t) = \frac{1}{2}\varphi^{-1}(\alpha t)$, $\alpha \in]0, 2/3]$.

Theorem

- If $\{\Omega_1, \Omega_2\}$ has a $\varphi-$transversality property at $(\bar{x}, \bar{y})$, then F has the corresponding $\varphi-$regularity property at $(\bar{x}, \bar{y})$.
- Let $\varphi(t) \geq t$ for all $t \in]0, \delta[$. If F has a $\varphi-$regularity at $(\bar{x}, \bar{y})$, then $\{\Omega_1, \Omega_2\}$ has the corresponding $\psi-$transversality at $(\bar{x}, \bar{y})$.

References

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THANK
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